Algebraic Topology

a Biography

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Les mathématiques sont une question de groupes.

Henri ***Poincaré***

On Geometry……
General Topology……
Paths……
Differential geometry……
Hyperspace……
Embedding……
Homology……
Cohomology……
Knot theory……
Open problems……

Geometry is an ancient part of mathematics that concerns itself with shape. It was founded by the ancient like Greeks to study shapes. The word Geometry itself is from two Greek words $\gamma\epsilon o$ and $\mu\epsilon\tau\rho o\varsigma$ so the word geometry literarily means measure of earth. The early rigorous system of geometry was Euclidian geometry known as classical geometry. It has seen through ages a lot of transformations. Today we see many new parts:

Analytic Geometry
Projective Geometry
Descriptive Geometry
Differential Geometry
Non-Euclidian Geometry
Algebraic Geometry
Differential Topology
Algebraic Topology
Geometric topology
Symplectic Geometry
Geometric topology…

In every instance, came a mathematician with a new idea and studied geometry by a new point of view and thus founding a new branch of geometry. Topology is just a new branch of geometry that was

founded in this way. Here we expose the algebraic tools devised principally by French scientist Henri **Poincaré**$(1854-1912)$ in a series of paper$(1895-1904)$ to found a new part of modern Geometry. Let's begin by the idea:

Let E be an ellipse of the equation $x^2+\frac{y^2}{a^2}=1$, geometrically (area, perimeter, distance, curvature) is different from the circle which we note S^1 of the equation $x^2+y^2=1$ We can deform the ellipse to the circle by putting $\frac{y}{a}=y'$, so we get (a function f) $y=ay$ or a transformation $y\longrightarrow ay'$ or more $(x,y)\longrightarrow(x',y')$ where $(x'=x,y'=ay)$ and.

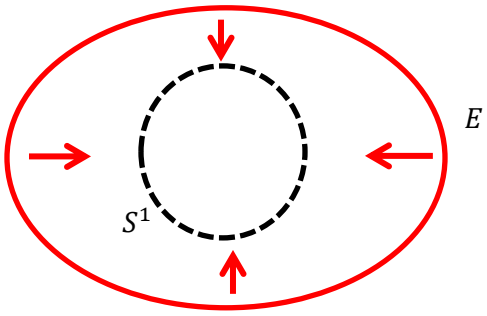


Fig. an ellipse to a circle.

We notice that the transformation f is *continuous*. A numerical($\mathbb{R}$ from to $\mathbb{R}$) function f is continuous at x_0 if

$$\lim_{x\longrightarrow x_0} f(x)=f(x_0)$$

Poincaré

or if $\lim_{\Delta y\longrightarrow 0}\Delta y=0$. when f is bijective, and both f and f^{-1} are continuous , as in our case, we say that f is called a *homeomorphism*[1]. So

there is a homeomorphism between a circle S^1 and an ellipse E in the plane and we say that they are *homeomorphic* or they are *topologically equivalent* and we write $E\cong S^1$. it is in a funny way people say that a topologist cannot say the difference between a cup and a doughnut. The idea of **Poincaré** is to classify shapes by homeomorphism between them. In this domain, we "stretch", "bend" and "twist" but

[1] From homeo and morphism " *form*".

we are not allowed to "tear" , or "glue[2]"(to identify distinct points) the shape. **Poincaré**'s aim was to discuss differential equations of higher order.

A method that would allow us to understand qualitative relationships in space of more than 3 dimensions could, to a certain extent, render services analogous to those rendered by figures. This method can only be Analysis Situs in more than 3 dimensions. Nevertheless, this branch of science has hitherto been little cultivated. After Riemann came Betti, who introduced some fundamental notions; but Betti was not followed by anyone. As for me, all the various paths I had successively embarked upon led me to Analysis Situs. I needed the data from this science to pursue my studies on the curves defined by differential equations and to extend them to higher-order differential equations and, in particular, to those of the three-body problem.

After all curves, surfaces and general manifolds are usually by differential equations. He set a program for that, but Dutch mathematician Luitzin **Brouwer**$(1881-1966)$ proved in 1911 an amazing theorem

Theorem:

If $n \neq m$ then there is no homeomorphism between $\mathbb{R}^n$ and $\mathbb{R}^m$.

It is called the *topology invariance*. This is an amazing result, Although there is a bijection f between $\mathbb{R}^n$ and $\mathbb{R}^m$ proved by German mathematician Georg **Cantor**$(1845-1918)$ in 1874 it is impossible to find a continuous bijection between them. So there is no homeomorphism between a cube and a square. After this, many work followed that resulted in a new part of geometry called *combinatorial topology* or *algebraic topology* , a marvelous part of research and with his celebrated *Poincaré conjecture*(1904) proved only in 2003. To delve in this new theory we need general topology and group theory. So algebraic topology uses groups to invest gate the proprieties of topological spaces and distinguishes between them. At first topology began combinatorial then became algebraic.

Algebraic topology is called *Analysis Situs*(analysis of position) before 1930. The name combinatorial topology is also known, it comes from the fact of counting the number of edges and faces… and not measuring. Sometimes it is simply called topology. Many early topology textbooks in XIX century used the word Topology to mean algebraic topology. then introduction to general topology French mathematician Jacque **Dixmier**$(1924-)$ says in his book

Topology divides into' general topology'(of which this course exposes the rudiments) and 'algebraic topology', which is based on general topology but makes uses of a lot of algebra.

Some early books on General topology called also Point-Set Topology.

$-$ *Grundzüge der Mengenlehre* (1914), by German Felix **Haudaurf**$(1868-1942)$.
$-$*Foundation of Point-Set Theory* (1932) by American Robert **Moore**$(1882-1974)$.

[2] Gluing and surgery are used in topology to solve problems.

—*Introduction to General Topology* (1934) by Polish Wacław **Sierpiński**(1882 − 1969).
—*Elements of the Topology of Plane Sets of Points* (1939) by English Maxwell **Newman**(1897 − 1984).
— *General Topology* (1955) , by American John **Kelly**(1916 − 1999).
—*Elementary General Topology* (1964) by American Theral **Moore**(1927 − 2023).

About algebraic topology

—*Analysis Situs* (1921) by American Oswald **Veblen**(1880 − 1960).
— *Vorlesungen über Topologie* (1923) ,by Hungarian Béla **Kerékjártó**(1898 − 1946).
— *Topology* in (1930) of Russian Solomon **Lefschetz**(1884 − 1972) ,then the second edition under the title Algebraic Topology (1942)[3].
—*An Introducton to Combinatory Topology*(1932) by Russian Pavel **Alexandrov**(1896 − 1982).
—*Elementary Concepts of Topology*(1932) by German Kurt **Reidemeister**(1893 − 1971).
—*Vorlesungen über die Theorie der Polyeder unter Einschluss der Elemente der Topologie*(1934) by Germans Enst **Steintz**(1871 − 1928) and Hans **Rademacher**(1892 − 1969).
—Leberbuch der *Topologie*(1934) by Germans Herbert **Seifert**(1907 − 1996) and William **Threlfall**(1888 − 1949).
—*Topology*(1935) by Russian Pavel **Alexandrov** and German Heinz **Hopf**(1894 − 1971).
— *Combinatory Topology*(1946) by Russian Pavel **Alexandrov**.
—*Fundation of Combinatorial Topology*(1954) by Lev **Pontriagin**(1884 − 1972).
—*An Introduction to Algebraic Topology*(1957) by Andrew **Wallace**(1926 − 2008).
— *Topology* (1961) by American John **Hocking** (1920 − 2011)and Gail **Young** (1915 − 2000).
—*Introduction to Topology* (1962) by American Bert **Mendlson**(1926 − 1988).
— *Topologie*(1964) by German Horst **Schubert**(1919 − 2001).

there is a homeomorphism between any two closed intervals, and there is a homeomorphism between any two open intervals for example $f(x) = cotan(x)$ is a homeomorphism between $]0, \pi[$ and $]0, +\infty[$

[3] Printed also by the American Mathematical Society.

Fig. unit ball $x^2 + y^2 + z^2 \leq 1$ is homeomorphic to a cube.

Fig. a line is homeomorphic to a path

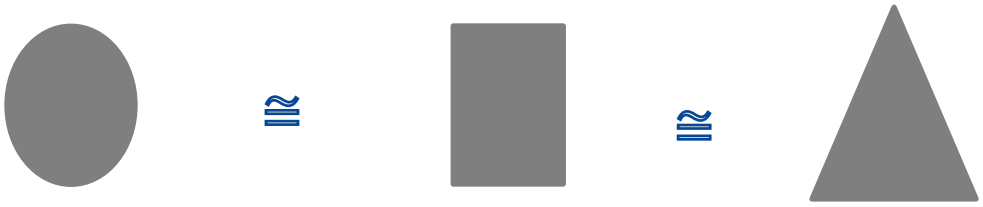


Fig. a disk is homeomorphic to a square and to a triangle.

Exercises

1) Show that $x,\ x^2,\ log(x)$ are continuous for every $x \in \mathbb{R}$.
1) Show that $f(x) = c$ is continuous for all x (by two methods).
1) Show that there is a homeomorphism between $\mathbb{R}^n$ and $\mathbb{R}^n$.
1)write the transformation equations between the two ellipses $x^2 + \frac{y^2}{5} = 1$ and $\frac{x^2}{11} + y^2 = 1$
1)write the transformation equations between the two circles $x^2 + y^2 = 2$ and $x^2 + y^2 = 7$
1) is there a bijection between even and odd natural numbers? is there a bijection between $\mathbb{N}$ and $\mathbb{Z}$?
1) show that there is a homeomorphism between $[a, b]$ and $[c, d]$.
1) show that the following are homeomorphism:
$-f(x) = \frac{1}{1-x}$ between $[0,1[$ and $[1, +\infty[$.
$-f(x) = \frac{x-a}{b-a}$ between $]a, b[$ and $]0,1[$.

General Topology

The word *topology* $\tau o \pi o \varsigma$ and $\lambda o \gamma o \varsigma$ study means *the study of position* , it was invented by German mathematician Johann **Listing**$(1808-1882)$ in 1847 in his book *Vorstudien zur Topologie*. the contained in this book are a part of algebraic topology . in our :

- Topology of X is the proprieties of a space X that do not change by continuous deformation like *compactness*…

-the topology of a surface . for example a sphere($S^2 \subset \mathbb{R}^3$) has a topology that if go from a point we return to it not as $\mathbb{R}^3$. S^2 has no *boundary* because there is a bijection between S^2 and the plan(stereotype projection). At the same time, the topology of sphere is different from that of a torus and there is no homeomorphism between them.

The idea of continuous deformation as a homeomorphism between two figures has been considered before **Poincaré** in :

- *Theorie der elementaren Verwandtschaft*(1863) by German mathematician August **Mobius**$(1790-1868)$.
- *Sur la déformation des Surfaces* (1866) by French Camille **Jordan**$(1838-1922)$.

Poincaré introduced his new ideas paper in 1895 and some mistakes were spotted by Danish mathematician Poul **Heegaard** $(1871-1948)$and German mathematician Walter **Dyck**$(1856-1934)$. He added in the period $1899-1904$ some supplements to correct an developed new ideas, known as *the five supplements*. So Poincaré work can be retraced as follows:

—Sur l'Analysis Situs(1892)
—Sur la généralization d'un theorem d'Euler relative aux polyèdres(1893)
—Analysis Situs(1895)
— first supplement :Sur les nombres de Betti(1899)
—second supplement (1899)
—third supplement (1900)

—fourth supplement (1904)
—fifth supplement (1904)

◯ *Topology on a Set:*

The subjects of general topology abound[4] and we restrict ourselves here to the minimum needed in algebraic topology. To define "continuity", "closeness" , " convergence",…general (or point set) topology is a part of algebraic topology and has become an introduction to many of mathematics subject as real analysis and function analysis as it deals with the topological proprieties of the $\mathbb{R}^n$. Let E be a non-empty set, with subsets $\tau \subset E$ such that

$i)$ $E, \emptyset \in \tau$
$ii)$ if $A_1, \dots, A_n$ then $\bigcap$ $A \in \tau$
$iii)$ if then $\bigcup$ $A \in \tau$

we say also that (E, τ) or simply E is a *topological space* . The set $\emptyset$ is open. A subset A is open : $\forall x \in A, \mathcal{O}: x \in \mathcal{O} \subset A$. Openness comes from limit of variables in analysis. in the *usual* topology $\mathbb{R}$ is open $\mathbb{R} =]+\infty, -\infty[$. We say that $A \subset E$ is the *neighborhood* of x if there is an open set $\mathcal{O}$ such that $x \in \mathcal{O} \subset A$.

the subsets are *open*. An open set(as an open interval $]a, b[$) is the neighborhood of all its elements. The more open sets two points both belong to, the closer they are. $]a, b[$ is an open set and $[a, b]$a closed whereas $[a, b[$ is neither open nor closed. what is a closed set ? a *closed* set $A \subset E$ is the complementary $\bar{A}$ of an open in E. E and $\emptyset$ are always open and closed. In a discrete topology τ_d, every set is open and closed. It turns out that we can define a topology τ in terms of closed subsets.

we can indicate the biggest topology and smallest topology on a set E. The first is the *discrete* topology, it is just simply $\tau = \mathcal{P}(E)$, the second $\{E, \emptyset\}$ the *indiscrete* topology. We say that at topology *finer* if it contain more open sets $\{E, \emptyset\} \subset \mathcal{P}(E)$.

A set A can be open in a topology τ_1 and not in another τ_2, here we real topology $(\mathbb{R})$.

In a topological space we can have a basis that every in we can it as a union of the basis. Every discrete topology τ_d has a basis they are the *singletons*. If the is countable we say that is *second countable*. The spaces $\mathbb{R}^n$ are second countable for their usual topology.

◯ *Continuity:*

In all this the idea is *paths* and *loop*. A path can be like

[4] Se book specialized in this domain like : *Topology* by James **Munkres**$(1930-)$, *General Topology* by John **Kelly**$(1916-1999)$ and *Topology* by James **Dugundji**$(1919-1985)$.

In this new language of point-set topology, the continuity of becomes : $f: (E, \tau_1) \longrightarrow (F, \tau_2)$ is continues if

$$f^{-1}(A \in \tau_2) \in \tau_1$$

The inverse image of $B \subset F$ always exists . We *connectedness* or *connectivity* We say h is connected if it is . is there a continuous $f : \mathbb{R} \longrightarrow S^0$

Homeomorphism

Let $f : E \longrightarrow F$ be a map, if f and f^{-1} are continuous then f is *homeomorphism* and E and F are called *homeomorphic* sets(or spaces). Very strange Brouwer theorem, there is no harm to remind you about it again

Theorem:

If $n \neq m$, then there is no homeomorphism between $\mathbb{R}^n$ and $\mathbb{R}^m$.

The theorem does not say that there may not be a continuous map f from $\mathbb{R}^n$ to $\mathbb{R}^m$ $(n \neq m)$, but it says that f^{-1} cannot be continuous. We can name two homeomorphic spaces by "topologically equivalent[5]".

Examples:

-the circle S^1 and the shape $\mathbf{8}$ are not homeomorphic.
-the interval $[a, b]$ and $]a, b[$ cannot be homeomorphic.

◯ *Separated topology:*

We say that if a topology τ is Hausdaurf(also called *separated* or T_2) if $x, y \in E$, $\exists V_x, V_y$, $V_x \cap V_y = \emptyset$. Every two points are separated by two distinct neighborhoods, there are some *pathologic* spaces that are not T_2. The topological space $\tau = \mathcal{P}(E)$ cannot be Hausdaurf.

◯ *Connected space:*

[5] We will see that two topologically equivalent have the same Euler number.

A set E is *connected* if it is not the union of two or more open disjoint sets. so E is *disconected* if $E = A \cup B$ and $A \cap B = \emptyset$. The sets $\mathbb{R}, \mathbb{R}^2, \dots, \mathbb{R}^n$ are connected. In set topogy langauge, E is conneted if the *only* subset open and close at the same time are E and $\emptyset$.

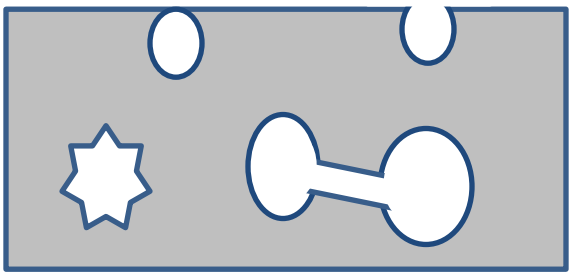


Fig. connected set in $\mathbb{R}^2$.

A surface E for example is *path-connected* if any two points in E can be linked by a curve that is entirely contained in E. Every convex set is arcwise connected, and evry arcwise connected is path connected . Every path connected space is connected the converse is not true.

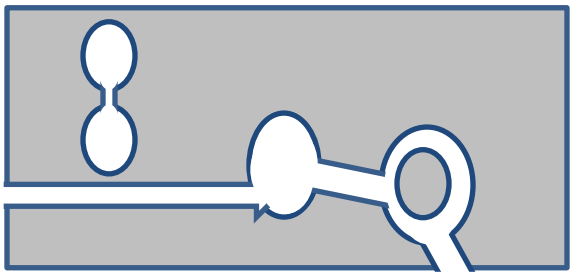


Fig. disconnected set in $\mathbb{R}^2$.

Two sets E and F one connected and the other disconnected cannot be homeomorphic. If we take any point from an open interval it becomes disconnected, unlike a closed interval. This is called *disjoint union* of spaces and noted

$$\coprod_{i=1}^{n} A_i$$

◯ *Compactness:*

The sphere S^2 is compact it is closed and bounded in $\mathbb{R}^3$. a compact every cover of X has a finite subcover. For example $]0,1]$ has the cover $\bigcup\ ]\frac{1}{n},1]$ we cannot extract a finite subcover. if $X=\mathbb{R}^n$ we have the Heine–Borel theorem

$\mathbb{R}$ is not compact, but $\mathbb{R}\cup\infty=\overline{\mathbb{R}}$ is compact. $\mathbb{R}$ is *locally compact* because every $x\in\mathbb{R}$ has a compact neighborhood.

◯ *Quotient Space :*

With an equivalence relation $\sim$, we can construct a new topological space $X/\sim$ from another topological space X called the *quotient space*. So we divide X in classes which are the elements of the new space. we will see for example the sphere S^n is quotient space : $D^n/\partial D^n$. One can verify that makes a *partition* of X.

Exercises

1) Show that $E-F=E\cap\bar{F}$.
1) tell why in a topology τ on a set E, $\cap$ is finite.
1) show that $\emptyset$ is a neighborhood for every x.
1) what is $\bigcap_1^\infty]-1,\frac{1}{n}[\quad,\ \bigcap_1^\infty]-\frac{1}{n},\frac{1}{n}[\quad,\quad\bigcap_1^\infty]-n,n[$?
1) what is the difference between $\bigcup_1^\infty[\frac{1}{n},1]$ and $\bigcup_1^\infty]\frac{1}{n},1]$?
1) show that $\tau_1\cap\tau_2$ is a topology , what about $\tau_1\cup\tau_2$?
1) let $E=\{x,y\}$ give all possible topologies on E? Are they Hausdaurf?
1) show that an indiscrete topology having two or more elements cannot be separated. Show that it is not metrizable.
1) is the set $\{\emptyset,\mathbb{R},all\ ]a,b[\}$ a topological space over $\mathbb{R}$?
1) is the set $\{\emptyset,\mathbb{R},all\ ]-a,a[\}$ a topological space over $\mathbb{R}$?
1) let E be a set and P a partition of E, is $\{E,\emptyset,P\}$ a topological space ?
1) Show that $\mathbb{N}$ and $\mathbb{Z}$ are closed in $\mathbb{R}$.
1) let $f:E\longrightarrow F$. Find $f^{-1}(\emptyset)$ and $f^{-1}(F)$.
1) show that if $\tau_2\subset\tau_1$ (τ_1 finer than τ_2) then $id_E:(E,\tau_1)\to(E,\tau_2)$ is continuous.
1) can we find a homeomorphism between an ellipse $x^2+\frac{y^2}{5}=1$ and hyperbola $x^2-y^2=1$.
1) Show that $E=\{x\}$ is a disconnected space.

Paths

We say that a continuous map $\gamma(t) : [a,b] \longrightarrow E,$ is a path in E . we can change the interval $[a,b]$ by scaling to $[0,1]$. Usually we use $[0,1]$ for simplicity. $\gamma(0)$ is the *initial* point and $\gamma(1)$ is the *final* or *end* point.

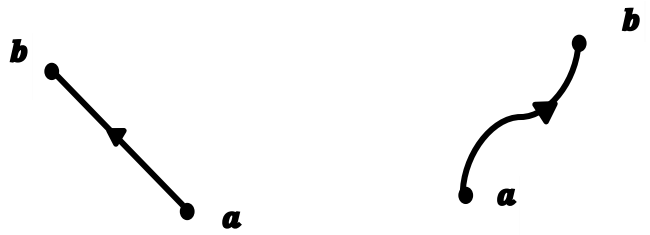


Fig. two paths in $E = \mathbb{R}^2$

a path can be in $\mathbb{R}^n$ as $\mathbb{R}^1, \mathbb{R}^2, \mathbb{R}^3$. It is not necessary injective. The inverse of a path $\gamma(t)$ is $\gamma^{-1}(t) = \gamma(1-t)$. Note that as $0 \leq t \leq 1$, γ^{-1} goes from $\gamma(1)$ to $\gamma(0)$.

Examples:

– is $\gamma(t) = (t+1, t-1)$ is a line in the plan $\mathbb{R}^2$.
– a path filling the space is the *Peano curve* , it is a path discovered by Italian Giuseppe **Peano**$(1858 - 1932)$ in 1890. Such a curve passes by every point in unit square. It is not injective(one to one).

Loops:

A special path is a closed path or a *loop*. We say that a path $\gamma(t) : [0,1] \longrightarrow E$ is a *loop* if : $\gamma(0) = \gamma(1)$

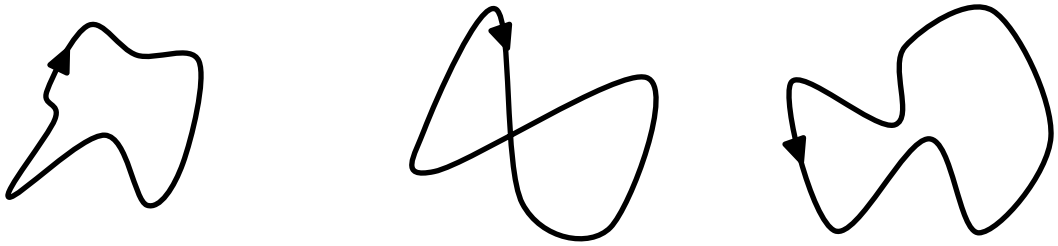


Fig. Loops in $X = \mathbb{R}^2$

Example is $\gamma(t) = (cos(t), sin(t))$ is a circle in the plan. A circle is a vey special loop. Interesting paths occur in $\mathbb{C}$ (which is $\mathbb{R}^2$). French mathematician Augstin **Cauchy**$(1789 - 1857)$ showed(1825) that if the function $f(z)$ is *holomorphic* (analytic) in a domain $D \subset \mathbb{C}$, then for any loop $\gamma \subset D$ we have $\int_\gamma f(z)dz = 0$. If there f has a singularity(poles) z_0 in D then the integral does not vanish, the value then has a relation with residues of the function

$$\int_\gamma f(z)dz = 2\pi i \sum_{i=1}^{n} res_i$$

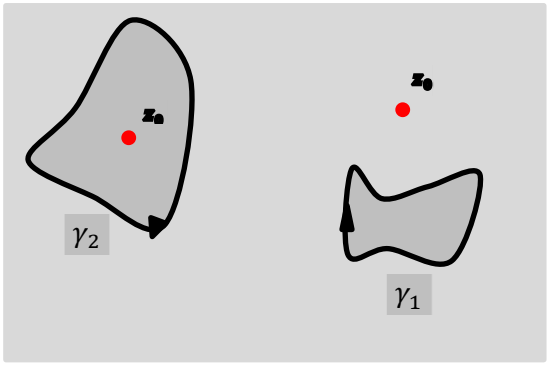


Fig. z_0 is a singularity of f.

If f has a pole at z_0 which means that $f(z) = \frac{g(z)}{z - z_0}$ then the residue at z_0 is $g(z_0)$. Let $f(z = x + iy) = u + iv$ so f is a map between two plan one (x, y) and (u, v). It is a transformation between two planes. We find a functional relation $y = f(x)$ and between , $v : v = g(u)$. A function $f(z)$ a *conform* transformation if it preserves angles. Two lines are parallel in (x, y) are also parallel in (u, v). Some maps are *isometries* , they preserve length and angles.

◯ *Homotopy:*

We say that two paths γ_1 to γ_2 are *homotopic* (homo same and topos place) if we can deform one in the other continuously without breaking.

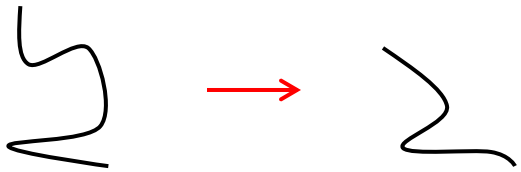


Fig.

Examples:

-We can form a homeomorphism between $[0,1]$ and any interval $[a,b]$ by the $y=\frac{x-a}{b-a}$.

We have seen homeomorphisms, In what follows ,we will see :

-a homotopy.
-homotopic to…
-retraction
-a homotopy type.
-a homotopy equivalence.
-a null-homotopy map.

We say that a path γ_1 is *homotopic* to another γ_2 if there is a continuous transformation H that transforms γ_1 to γ_2, continuously(which means that it is continuous) so how to find ? we need time to so And H is defined as

$$H : X \times [0,1] \longrightarrow Y$$

and the is

$$H(x,0) = \gamma_1$$
$$H(x,1) = \gamma_2$$

and we note the homotopy by $\gamma_1 \cong \gamma_2$, we say that H is a *homotopy* between γ_1 and γ_2.

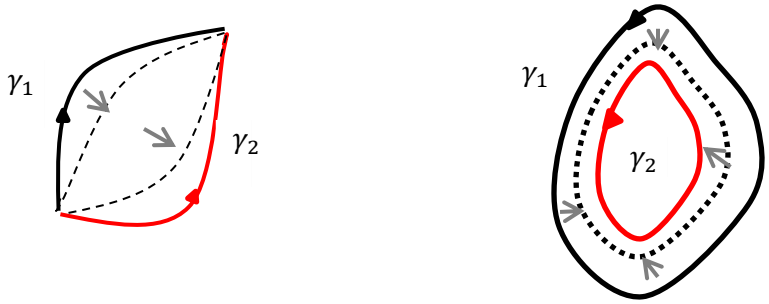
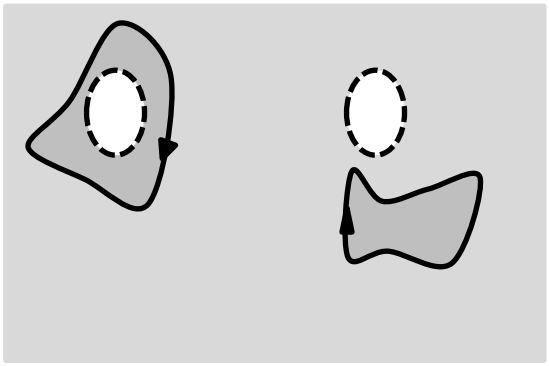


Fig. two homotopic paths.

A path is just a function. So two function f and g are homotopic $f \cong g$. The Möbius[6] band is homotopic to an usual band and to a circle. A *homotopy* (or homotopy of paths) is a family of continuous functions f_i. A to an intermediary path between γ_1 and γ_2.

$$H = fg(t) = \begin{cases} f_{2t} & 0 \le t \le \frac{1}{2} \\ f_{2t-1} & \frac{1}{2} \le t \le 1 \end{cases}$$

Sometimes we consider the variable t as *time* because we see the transformation as a movement. a continuous map is *null-homotopy* map if it is homotopic to a constant map.



$\mathbb{D}$

Fig. two paths cannot be homotopic in D.

[6] Mobius band has *chirality*

so γ_1 to γ_2 cannot be homotopic and we write $\gamma_1 \ncong \gamma_2$. Another concept is *isotopy* a deformation of a map without self-intersecting.

Baer theorem(1928):

two simple curves on a $2-$manifold are isotopic iff they are homotopic.

We have seen connected, path connected simply connected. A domain D in $\mathbb{R}^2$ is *simply connected* if we can deform any loop in D continuously to point in D. Group is A path

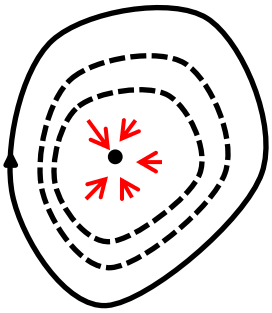


Fig. to a point

A form is $\int_C Pdx + Qdy = 0$ if $P_y = Q_x$. The circle S^1 as a space is not simply connected.

Definition:

a space X is *simply connected* if homotpic to a point in X.

Retraction:

A map $f : X \longrightarrow Y$ is called a *constant* map if $f(x) = a$. A *deformation retraction* is a homotopy which we precise now.

Definition

a deformation retraction is a deformation of X to $A \subset X$ and keep A intact.

We can formalize by a map $f: X \longrightarrow X$,

There is a difference between retraction and deformation retraction. A deformation retraction is a special case of retraction, it is a family with time t. If X is homotopic to a point, we say that X is *contractible*.

Examples:

–the annulus retracts to a circle.
–a Mobius band retracts to a circle.
–S^1 Mobius is not a retract of D^2.

We will see that $\pi_1(E)$ is the group of equivalence classes of closed paths based on a fixed point x_0 , it's the fundamental group. A contractible space is simply connected and we will see that it has a trivial group.

As we know a deformation retraction is a special case of a homotopy. The circle S^1 and cylinder $S^1 \times [0,1]$ are retraction but not homeomorphic.

Examples:

–Any convex of $\mathbb{R}^1$ is contractible.
– $\mathbb{R}^n$ is contractible and simply connected.
–For $n \geq 1$, S^n is not contractible, a circle cannot be deformed within itself to a point.

Homotopy type:

So X, Y have the same type of homotopy if H exists. We will see

We now define the *homotopy type*.

Definition

A space with the homotopy type of a point is *contractible*.
-any homeomorphic spaces are of the same homotopy type.

Examples:

-homotopy type of circle S^1 is that of an annulus or punctured plane.

Theorem

X and Y Homeomorphic $\Longrightarrow$ have the same type of homotopy

X and Y have the same type of homotopy $\nRightarrow$ X and Y are homeomorphic

Homotopy equivalence.

Two spaces X and Y

Definition

X and Y are *homotopy equivalent* if there exist two homotopies $f: X \longrightarrow Y$ and $g : Y \longrightarrow X$ such that $fg = \mathbb{1}_Y$ and $gf = \mathbb{1}_X$, the identity maps.

It is not necessary that f is the inverse of g. Every homeomorphism is a homotopy equivalence. Two homotopy equivalent spaces have isomorphic groups, the inverse is false.

Examples:

– The annulus has the homotopy equivalence of S^1.
-Homotopy equivalence is weaker than homeomorphism.

To feel the subtle difference between homeomorprism and homotopy equivalence, note that the following spaces are homotopy equivalent but they are not homeomorphic

Fig. are homotopy equivalent but they are not homeomorphic

No one is the deformation retract of the other. Every homomorphism is deformation retract and every deformation retract is an homotopy equivalence.

A *homotopy sphere* is a topological space that is homotopy equivalent to the n −sphere.

Remark that paths can be put in equivalence classes. So we speak of equivalent classes $[\gamma_1] \sim [\gamma_2]$. We have first to define the law of composition between paths. then to show that it is *reflexive*, *symmetric* and *transitive* so homotpy is an equivalence relation.

Theorem
The homotopy of paths is an equivalence relation.

-The plane has only one homotopy equivalence.

- $\gamma_1 \sim \gamma_2$
- $\gamma_1 \sim \gamma_2$ then $\gamma_2 \sim \gamma_1$
- $\gamma_1 \sim \gamma_2$ and $\gamma_2 \sim \gamma_3$ then $\gamma_1 \sim \gamma_3$

in this case : $\gamma_1(1) = \gamma_2(0)$.

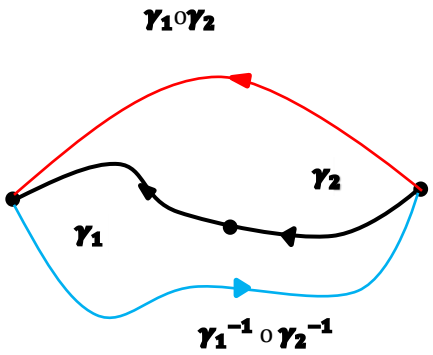


Fig. product of two paths.

In the previous we remark $\gamma_1(1) = \gamma_2(0)$. We the same operation with loops

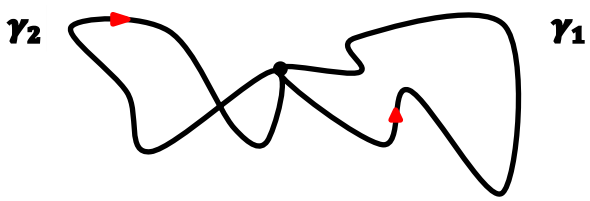


Fig. a product of two loops.

We will see that the fundamental group is a set of special path equivalences γ that verifies the axioms of a group. The equivalence relation of general paths cannot be a group.

◯ *What is Algebraic Topology:*

Algebraic topology is a part of geometry that use abstract algebra(group,…) to classify topological spaces. It has changed a lot since **Poincaré**. French mathematician Jean **Leray**$(1906-1998)$ said in 1953:

…We no longer speak of Analysis Situs, but of algebraic topology. Its objective no longer seems to be to reveal the qualitative relations of space, but to attach quantitative, or more precisely, algebraic notions to qualitative, or more accurately, topological data. Today, Poincaré's terminology is outdated; he himself had to correct his initial definitions. Each subsequent advance required a revision of the vocabulary and basic concepts. Algebraic topology is constantly changing its purpose, methods, and language. Just as a philologist cannot ignore Indo-European roots, a topologist must be familiar with Poincaré's memoirs.

The objectives of algebraic topology:

–it how to assign to each space invariants in order to distinguish between the spaces It searches for invariant in topological spaces:

∗ connectedness(topo)
∗ compactness(topo)
∗ Euler characteristic(topo)
∗ Betti numbers(topo)
∗ fundamental group(algebraic)
∗ homology group(algebraic)
∗ cohomology group(algebraic

– at the same time look for invariants are relatively easy to compute.

–one of the aims of algebraic topology is to find *holes* in surfaces. Some define algebraic topology discovers n-dimensional manifolds and characterize them by means of invariants of topological spaces.

–Also the tools to distinguish between knots and find out if two knots are the same.

Exercises

1) what does $f(z) = z^2$ do to the points z in $\mathbb{C}(\mathbb{R}^2)$?

1) Let $f(z) = z^2$. Find the transformation of lines $x = const$, $y = const$.

1) using residue theorem show that $f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0} dz$.

1) Are the following connected ? simply connected ?

* the annulus $2 < |z| < 4$
* $|z| > 5$
* $|z| < 1$
* $\Re(z) > -2$

1) Show that $\gamma(t) = (2t, 3t)$ is a line in the plan.

1) Show that $\gamma(t) = (\cos(t), \sin(t))$ is a circle.

1) Show that $\gamma(t) = (1 + t, 1 - t)$ is a line.

1) show that $f(x) = \frac{x}{1+x}$ is a homeomorphism between $]-\infty, -1[$ and $]+1, +\infty]$.

1) show that $f(x) = \frac{x}{1-x}$ is a homeomorphism between $]\frac{1}{2}, +\infty[$ and $]-1, 1]$.

1) are an interval I and a point x homotopy equivalente?

1) show that lot of relations are of equivalence : $=$ in $\mathbb{R}^n, \mathbb{C}, \ldots$, // in $\mathbb{R}^2$, $\equiv$ in $\mathbb{Z}_n$, …

Differential Geometry

Differential geometry is the use of calculus apparatus to study curves and surfaces. A *curve* is a vector valued function. It can be plan (in $\mathbb{R}^2$) of the implicit equation $F(x, y) = 0$, and a functional relation

between y and x can exist according to the *implicit theorem*.

Theorem

Let $F(x,y)=0$ be an implicit relation between x and y. Then F can locally define a numerical function $y=f(x)$ at $x=x_0$ if $\frac{\partial F}{\partial y}\neq 0$.

The condition $\frac{\partial F}{\partial y}\neq 0$ is in the place of the Jacobian.

A curve can be spatial (in $\mathbb{R}^3$, in French *courbe gauche*) , it can be also the intersection of two surfaces. For simplicity, a curve is given by a parameter t. This time from mechanics the itinerary of a particle is by coordinates: x(t),y(t) and z(t). In $\mathbb{R}^2$ a curve can be seen $\gamma(t)=x(t)e_1+y(t)e_2$ and in $\mathbb{R}^3$ $\gamma(t)=x(t)e_1+y(t)e_2+z(t)e_3$.

Examples:

– the curve of the parameterization $\gamma(t)=cos(t)e_1+sin(t)e_2$ is a circle in $\mathbb{R}^2$
–the curve of the parameterization $\gamma(t)=cos(t)e_1+sin(t)e_2+be_3$ is a helix in $\mathbb{R}^3$

Arc length

A curve can be parameterized by its *arc length* noted by the letter s (as in the definition of the radian) $y=f(s)$. We use arc length parameterization (if it is possible) to simplify an underlying geometry of the curve, this makes tangents and curvature simpler to find. An *allowable* change of parameter. Does a curve always admit an arc-length parameterization? We can calculate area between two curves ,

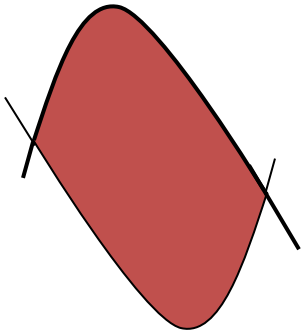


Fig. An area between two curves

We can for example calculate the length s of a curve between two points and the differential is

$$ds = \sqrt{dx^2 + dy^2}$$

where dx^2 means $(dx)^2$. The length between a and b is then $\int_a^b \sqrt{1 + f'(x)^2}dx$. For parametric curve

$$ds = \sqrt{(f')^2 + (g')^2}dt$$

in other when $\gamma(t)$ is vector $s = \int_a^b |\gamma'(t)|\, dt$, $\gamma'(t)$ is the tangent vector or the velocity.

A curve is ***rectifiable*** if its length is finite. The curve defined by $y = sin(\frac{1}{x})$ is not rectifiable on $]0, a]$.

Regular curves:

A curve $\gamma(t)$ is regular if $\gamma'(t) \neq 0$. In the case $\gamma'(t) = 0$ we cannot define many useful parameters like: curvature and tangent.

A curve presented by the equation $F(x, y) = 0$ is a parameterized if $\frac{\partial F}{\partial x} \neq 0$ and $\frac{\partial F}{\partial y} \neq 0$. The same equation $F(x, y, z) = 0$ is a parameterized if $\frac{\partial F}{\partial x} \neq 0$, $\frac{\partial F}{\partial y} \neq 0$ and $\frac{\partial F}{\partial z} \neq 0$.
can every curve be parameterized by its arc length?

Theorem

If C is regular then it can be parameterized by its arc length.

◯ *Curvature:*

An important propriety of curves and surfaces is the *curvature* noted κ. For a curve, the idea is illustrated by a point that moves along a curve and gives information how it bents as it goes, hence a relation between angle and length so the curvature[7]

[7] This has been accomplished by Euler around 1736.

$$\kappa \sim \frac{angle}{lenght} \sim \frac{\Delta\alpha}{\Delta l}$$

So it is a limit $\kappa = \lim_{\to 0} \frac{\Delta\alpha}{\Delta l}$. The curvature is a local property of the curve. It informs us how the length of the curve changes as the curve is deformed. At every point on the curve passes a circle of radius R that has the curvature $\frac{1}{R}$, where $R = \frac{1}{\kappa}$ is the radius of curvature. The radius circle R goes to infinity when κ to goes to 0.

For, when a curve is a circle, its curvature $\kappa = \frac{\alpha}{\alpha R}$ and the curvature of the circle is $\kappa = \frac{1}{R}$ it is constant. The curvature of a curve given by numerical function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is $\frac{|y''|}{(1+(y')^2)^{3/2}}$ it is the *scalar curvature*, it can have different curvature at different points .note that κ is positive[8]. We will see that in the case of general manifold the curvature can be a very complicated tensor. The direction vector

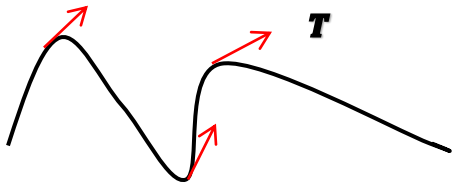


Fig. direction vector

Definition:

the curvature κ. Is the rate of change of the direction vector

The curvature as a vector measures how the tangent vector changes with respect the arc length s

$$\boldsymbol{\kappa} = \frac{dT}{ds}$$

with $\boldsymbol{\kappa} \perp T$. A curve has a curvature $\boldsymbol{\kappa}$ which is a vector such that $\kappa = |\boldsymbol{\kappa}|$ is a scalar.

The total curvature of a closed curve is always $2n\pi$

[8] This formula is due to **Newton**.

$$\int_a^b \kappa(t)dt = 2n\pi$$

Here we need the Jocabian.

Theorem:
A plane curve C in $\mathbb{R}^2$ is uniquely determined by its curvature κ.

the extremum for a curve can be either local or global. The point $x = 0$ is a global extremum for $f(x) = x^4$ whereas $x = 1$ is a local extremum for $f(x) = x^2 + x - 2$. In the plane we get what sometimes called the Frenet *dihedron*.

$$T, N$$

Torsion:

In $\mathbb{R}^3$ a space curve C is more complicated, in $\mathbb{R}^3$ it may have a new parameter called *torsion* and noted τ. on a point of a curve we can three vectors at a point :T(tangent), B(binormal) and N(normal) with

$$B = T \times N$$

The trihedron.

If C is a plane in $\mathbb{R}^3$ then $\tau = 0$. Frenet conditions due to French mathematician Jean **Frenet**$(1816 - 1900)$. An important result in theory of curves is that

Theorem:
A *regular* curve is *uniquely determined* by its curvature κ and *torsion* τ.

In $\mathbb{R}^3$ we three : T , B and N. Through the known Frenet–Serret equations

$$\begin{cases} T' = \kappa N \\ N' = -\kappa T + \tau B \\ B' = -\tau N \end{cases}$$

By a matrix

$$\begin{bmatrix} a \\ g \\ g \end{bmatrix} = \begin{bmatrix} 0 & & 0 \\ & 0 & \kappa \\ 0 & \kappa & 0 \end{bmatrix} \begin{bmatrix} a \\ g \\ g \end{bmatrix}$$

There are $\kappa =$ constant 0 plane positive sphere, and negative saddle. So we can define the curvature by

$$\frac{dT}{ds}$$

A curve ($\tau = 0$) can be found by integrating the Frenet equation.
-a circular helix is a space curve with constant κ and τ.

<u>Surfaces:</u>

A surface can be by the equation is $z = f(x, y)$ or more general $f(x, y, z) = 0.$ a face unit on a surface is called a *<u>patch</u>*. The area of a domain D on a surface $z = f(x, y)$ is $\iint_D \sqrt{1 + {f_x}^2 + {f_y}^2}\, dxdy$. The volume by a function in D is the double integral is $\iint_D f(x, y)\, dxdy$.
The curvature is a local property of the surface and the one-sidedness is a global one.
For a *<u>regular</u>* parameterized surface $F(x, y, z) = 0$ at a point if $\frac{\partial F}{\partial y} \neq 0$, $\frac{\partial F}{\partial y} \neq 0$ and $\frac{\partial F}{\partial z} \neq 0$.

A *<u>geodesic</u>* is the shortest path between two points:

−in a plane it is a line.
−on a sphere, lines cannot be the shortest path between two points. For each point and direction there is exactly one geodesic(PDE). The distance between two points on a sphere is the *<u>great circle</u>* joining them.
−on a general surface it is the proper “geodesic”.

If a surface $z = f(x, y)$ has an *<u>extremum</u>* (minimum or extremum) at the point (x_0, y_0) then $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0$, but the inverse is not true(hence the *<u>critical</u>* point), a counterexample is the saddle of the equation $z = x^2 - y^2$. As in the case of curve, the extremum can be either local or global.

-saddle
-the $z = x^2 + y^2$.

The gradient ∇f is orthogonal to the surface $f(x, y, z) = 0$. We first dr so the is $A \circ dr$ which gives $dr = dx + dy + dz$

Complex function led to interesting surfaces called *Riemann surface*[9]. An example is the complex plan $\mathbb{C}$ or any open set $U \subset \mathbb{C}$.

Tangent space:

We try to find vector tangent T. If $f(x)$ is numerical function then the tangent at a point x_0 is defined if $f'(x_0)$ exits and it equation is $y = f'(x_0)(x_0 - x) + f(x_0)$.

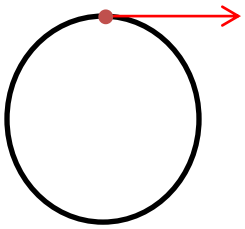


Fig. a tangent vector at

Take a circle $x(t) = rcos(t) + rsin(t)$ we that the tangent $x'(t)$ is perpendicular to the circle. Using the arc length, it is similar to the speed of a particle Find the expression $v = \frac{ds}{dt}$

Next to the derivative, we have *directional* derivative of f is the derivative of f in the direction of a vector v. For two variables

$$\lim_{h \to 0} \frac{f(x + hv, y + hv) - f(x, y)}{h}$$

or

$$\frac{\partial u}{\partial v} = \sum \frac{\partial u}{\partial x_i} cos(\alpha_i)$$

[9] Introduce by **Riemann** in his dissertation paper *"function of complex variables"*(1851) and published in 1867.

If $u(x,y) = x - y$ find the derivative in the direction of the vector $v = i + j$. So

$$\frac{\partial u}{\partial v} = 1\cos\left(\frac{\pi}{4}\right) - 1\cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0$$

In the case of numerical function it is just the ordinary derivative

$$\lim_{h\to 0} \frac{f(x+h1) - f(x)}{h}$$

Let u_n be the unity vector, we have the relation

$$\frac{\partial u}{\partial u_n} = |\nabla(u)|$$

We know that a tangent to a plane curve at a point x_0 is the which s and the equation $y = f'(x_0)(x - x_0) + f(x_0)$. A tangent line is by a ***vector*** tangent at a point p. $f'(x_0)$ is the number.

Tangent vector: if f is a numerical then the tangent vector at a point p, then the tangent T is the collection of all vectors tangent to p. Let γ be a regular curve , $\gamma'(t)$ is the tangent vector to M at p.

Tangent plane : in the case of a surface at a point p, is the vectors at point p is the collection of all vector tangents at in all directions.

The ***tangent space***: T_pM is a general concept at point p is the collection of all vector tangent at p. T_pM is a vector space, if $p \neq q$ then $T_pM \neq T_qM$.

In the domain of particle physics, the tangent space T_pM to a manifold M is viewed as the space of velocities of particles moving on M which is very illuminating.
The collection of tangent spaces is ***bundle*** :

$$TM = \bigcup_{i=1}^{n} T_{p_i}M$$

Principal curvature:

At appoint x_0 on a surface we can determine two directions so two principle curvatures (κ_1, κ_2) , κ_1 is positive(upward) and κ_2 negative(downward).

−The maximal curvature κ_1
−the minimal curvature κ_2

To find them to a point x, we the normal plane and surface at x which gives a curve the curvature of the curve at x is the first κ_1 and another plan perpendicular to the first plane the plan to . For example, the south pole of a sphere we have $\kappa_1 = \kappa_2 = 1$. The space $\mathbb{R}^n$ has $n-1$ principal curvatures $\kappa_1, \dots, \kappa_{n-1}$

We also define the *mean curvature* which is : $H = \frac{\kappa_1 + \kappa_2}{2}$.

Surface	κ_1	κ_2	H
plan	0	0	0
cylinder	0	1	$\frac{1}{2}$
sphere	1	1	1
Hyperbolic paraboloid	-1	1	0

Fig. curvatures of some shapes.

in other we can calculate the curvature of a surface $u(x, y) = 0$ by the Hessian matrix

$$H = \begin{bmatrix} u_{xx} & u_{xy} \\ u_{yx} & u_{yy} \end{bmatrix}$$

From Schwarz theorem the Hessian is a symmetric matrix, hence its eigenvalues are real. H can be used also to detect concavity and convexity of surfaces. The determinant of Hessian is $D = u_{xx}u_{yy} - (u_{xy})^2$. It gives on convexity and extremum points and curvature. For instance, f $D < 0$ at a point (x_0, y_0) then the point is a *saddle* point.

◯ *Vector fields:*

A vector field is a map : $\mathbb{R}^m \longrightarrow \mathbb{R}^n$, it associates to every point a vector. A vector field so it is of position used in geometry and bundles.

Example

- $xi + xyj$ is a vector field in the plan $\mathbb{R}^2$.
- $(t+1)i + tj$ is a vector field in the plan $\mathbb{R}^2$.
- $xi + yj + z^2k$ is a field in the plan $\mathbb{R}^3$.
- $(i + (t-1)j + (t^2 + t)k$ is a field in the plan $\mathbb{R}^3$.

In the domain of physics and mathematical physic there is the ***nabla*** operation[10] ∇ (called also ***del***) which is a derivation operation

$$\nabla = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}$$

With nabla we can define three principal operators: gradient (∇), the curl $(\nabla \times)$ and the divergence $(\nabla \circ)$. Let $u(x,y) = x^2 + yx$ be a (scalar) function of two variables then the gradient $\nabla u = (y + 2x)i + xj$ is a vector field. Let $u(x,y,z) = x^2 + yx + z^2$ be a function of three variables then the gradient $\nabla u = (y + 2x)i + xj + 2zk$ is a vector field. Also the $div\ grad f = \nabla \circ \nabla f = \nabla^2 f = \Delta f$.

We have

$$\textit{Scalar function} \xrightarrow{\nabla} \textit{vector field} \xrightarrow{\nabla\times} \textit{vector field} \xrightarrow{\nabla\circ} \textit{Scalar function}$$

Integration:

Let $A(x,y)$ a vector field then is $\int Adr$. the dr is just $xdx + ydy$ we can calculate in a path by changing variable $\int_a^b (t^2 + 2t)i + tj\ dt =$

Examples:

– The $\int_a^b (t^2 + 2t)i + tj$ we

◯ *Gauss geometry:*

Gauss was interested in the curvature of the Earth surface, he has already defined the distance between two points on a surface. the discovery made in his celebrated paper[11] in 1827. **Gauss** curvature of a surface K at a point is given in terms of two principal curvatures $K = \kappa_1 \kappa_2$. In fact, K , as in the case of scalar curvature, is the ratio of the solid angle subtended by the normal projection of a small patch divided by the area of that patch.

[10] Introduced by Peter **Tait**$(1831 - 1901)$ in 1867. the word *nabla* is in Greek , it means *harp* . The symbol looks like a harp.

[11] Under the title "*Disquisitiones generals circa superficies curvas*" or "*general investigation on curved surfaces*".

$$K = \frac{\alpha}{Area}$$

Gauss discovered that his curvature is ***intrinsic*** like the arc length and unlike the principal curvatures. Which means that we do not need to view the space from outside.

manifold	Gaussian curvature
S^2	$\frac{1}{R^2}$
plane	0
cylinder	0
Hyperbolic paraboloid	-1
hyperbolic hyperboloid	< 0
saddle	< 0
Elliptic hyperboloid	> 0
inside T^2	< 0

Fig. usual spaces with their Gauss curvature.

A sphere as viewed in $\mathbb{R}^3$ is curved, we can know that it is curved by drawing a triangle and ind that $\sum_1^3 \alpha_i > \pi$ without considering it as view as in $\mathbb{R}^3$, so the Gauss curvature for the sphere $K \neq 0$. Whereas a cylinder although is seen curved in $\mathbb{R}^3$ if we a draw triangle $\sum_1^3 \alpha_i = \pi$ like a plane, so its Gauss curvature $K = 0$.

We see that it is evaluated by measurements in the surface itself without outside considerations. This discovery named by Gauss *Theorema Egregum.*

-if $K < 0$ the shape is hyperbolic and the sum the angles of a triangle is $< \pi$. Like the saddle surface $z = xy$, it is a stationary point which is neither a maximum nor a minimum.
-if $K > 0$ the sum of the angles of a triangle is $> \pi$. The κ of circle S^1 and sphere S^2 and is positive and constant.
-if $K = 0$ the shape is parabolic, the sum of the angles of a triangle is $= \pi$.

The shape operator:

We can get the two curvatures κ_1 and κ_2 from the *shape operator*. It is defined as a linear operator on the tangent plane T_pX at the point , it indicates how M is curving at he point p.

$$S : T_pM \longrightarrow T_pM$$

We have the result

-the eigenvalues of S are the principles curvature κ_1, κ_2.
-the value $\frac{1}{2}Tr(S)$ is the mean curvature H.
-the determinant of S is the Gauss curvature K.

Another is the Gauss map

$$G: M \longrightarrow S^n$$

whose differential is the shape operator : $G = dS$.
-the Jacobian of the Gauss map is the Gauss curvature.

Let $z = F(x, y)$ a surface then the Hessian $\begin{bmatrix} F_{xx} & F_{yx} \\ F_{xy} & F_{yy} \end{bmatrix}$ is related to the Gauss curvature

$$K = \begin{vmatrix} F_{xx} & F_{yx} \\ F_{xy} & F_{yy} \end{vmatrix}$$

and the eigenvalues of H are the principles curvatures

$$\mathrm{II} = \begin{bmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{bmatrix}$$

A *bilinear form* called the fundamental form II. We have $tr(\mathrm{II}) = \kappa_1 + \kappa_2$ and $det(\mathrm{II}) = \kappa_1\kappa_2 = K$ the two fundamental forms

$$Fdx + dy \qquad , \qquad qdx + dy$$

The Gauss–Bonnet formula has a relation with Euler characteristic $\chi(X)$, so relates the topology ($\chi(X)$) of a surface to its curvature

Gauss–Bonnet theorem
Let K be the Gauss curvature then :

$$\iint_X K da = 2\pi\chi(X)$$

French mathematician Pierre **Bonet**$(1819-1892)$ published in 1848 a special of this theorem. It also defined as a trace of the matrix stem from the forms. We can by using matrices calculate K and H. There are spaces with constant K like the plane, the sphere.

We to minimize area of a surface and we define *minimal surface* as a surface whose mean curvature $H=\frac{\kappa_1+\kappa_2}{2}=0$ everywhere this means that it K curvature is negative. Like the plane and the *catenoid*.

Exercise

1) show that the length of an arc is αR.
1)Find the area and the volume of a sphere with $R=2cm$.
1)Calculate the curvature of a line in $y=-2x-1$.
1) show that the curvature at an inflexion point is 0.
1) Calculate the curvature of parabola $y=x^2+1$ at the points: $x=0,2,-2$.
1)Calculate the length of the curve $y=x+1$ in $[1,4]$.
1)find the equation of a line tangent to the curve $x(t)=2x^2-1$ at $x=3$.
1) let$f(x)=x+3$. Calculate the length s of the curve from $x=1$ to $x=2$.
1) let $f(x)=x+1$. Calculate the curvature of the curve at $x=2$.
1) find the area of the surface between the curve x^2+x-1 , the real axis and the two points $x=0,2$.
1)Calculate the surface area between $y=x^2-1$ and $y=-x^2+1$.
1)Let $x(t)$ the equation of a circle . Show that $x'(t)$ is perpendicular to $x(t)$.
1)Show that for $b>0$ the curve $y=\frac{1}{x}$ is not rectifiable in $]0,b]$.
1)calculate the Hessian for $f(x,y)=x^3+xy^2$.
1)Calculate the surface area of $y+z=-2x^2-1$ that lies in the rectangle $x=1,2$ and $y=0,3$.
1) show that if u is a unit vector function, then $\frac{du}{dt}$ is orthogonal to u.
1) let $f(x)=x+1$. Calculate the curvature of the curve at $x=2$.
1) is $(1-t)e_1+(3t^2-2)e_2$ a regular curve? Show that it is a parabola in the plane $\mathbb{R}^2$.
1) Find the unit vector orthogonal to the surface $x^2+z-y=0$ at $p(2,1,3)$.
1) define a tensor. A tensor has many directions.
1) define a direct sum(or direct product) of v_1 and v_2.
1) define a direct sum of V_1 and V_2
1) define a tensor product v_1 and v_2.
1) define a tensor product of V_1 and V_2
1)show that a minimal surface has a negative Gauss curvature.

Differential Forms

Differential (or exterior) forms are important in geometry and physics, and they are at the origin to the solution of very important problems in differential geometry is differential forms especially differential equation. This subject is a part of ***exterior algebra***[12]. To every vector field is associated a differential form.

Differentials

We know from calculus that $dy = f'(x)dx$ is a differential, when we have 2 variables the differential is $du = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$, du here is the *total* differential of f. In many occasions we find a differential equation of the form $\frac{dy}{dx} = \frac{P(x,y)}{Q(x,y)}$ which to $P(x,y)dx + Q(x,y)\,dy = 0$ (abstraction of sign). We say that the form $P(x,y)dx + Q(x,y)\,dy$ is *exact* if there exists $f(x,y) = c$ such that

$$d(x,y) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = P(x,y)dx + Q(x,y)\,dy = 0$$

which amounts to the condition

$$P_y(x,y) = Q_x(x,y)$$

Differential forms are classified according to its degree. It is what are the differential forms in $\mathbb{R}^1$, $\mathbb{R}^2$, $\mathbb{R}^3$?

After we have defined the tangent T_pX, what is a differential form of degree k?a form ω is defined on a tangent space the existence of forms depends on $\mathbb{R}^n$, we talk of *$k-$form* of degree k : $k \leq n$.

-in $\mathbb{R}$: $0-$ form is ta function $f(x)$ and 1- form is : $df = f'(x)dx$
-in $\mathbb{R}^2$: $0-$form is a function of two variables $f(x,y)$, $1-$ form is : $f_x dx + f_y dy$ and $1-$ form is : $adxdy$.

[12] A part of *exterior algebra* created by German Hermann **Grassmann**$(1809-1877)$ an developed further by French Elie **Cartan**$(1869-1951)$.

-in $\mathbb{R}^3$: $0-$form is a function of two variables $f(x,y,z)$ and $1-$ form is : $Ady+Bdy+C\,dz$, the 2-form is $Adxdy+Bdxdz+Cdydz$. The $3-$ form is : $Adydydz$.

You can add and subtract differential forms of the same degree. We can multiply a function $g\omega$. A differential form looks like vector fields. How to define a differential form ?

Definition

form is from to a number $\omega: T_p\mathbb{R}^n \longrightarrow \mathbb{R}^n$.

$$\omega : T_pM \longrightarrow \mathbb{R}$$

a differential form is a smooth, alternating and multilinear object that may be differentiated and integrated, …So put a vector v from T_PM in df giving a number and gives the instantaneous change of f in the direction of v like directional derivative.

A p-form is a multilinear skew symmetric map. We know that $det\,(A)$ is an alternating multilinear form. Both tensors and differential forms are used in solving in a lot of problems in physics and geometry.

which $ddx=0$, $dxdy=-dydx$,… we do not mix with facts about integration. The integration is an operation with $\int_a^b = -\int_b^a$. And double integrals : if rectangle : $[a,b]\times[c,d]$ then :

$$\int_a^b\int_c^d f(x,y)dxdy=\int_c^d\int_a^b f(x,y)dydx$$

The product of $k-$form and $l-$form is a $(k+l)-$form.

The exterior product:

Grassmann called it[13] "*außeres produkt*" (the cross product $\times$ is a special case). We define $\wedge$ as a matrix

$$\alpha\wedge\beta=\frac{1}{2}[v_1^Tv_2-v_2^Tv_2]$$

Then we define a new operation in $T_p\mathbb{R}^n$ by

$$\alpha\wedge\beta:\ T_p\mathbb{R}^n\longrightarrow\mathbb{R}^n$$

[13] In his book *Die Lineale Ausdehnungslehre* (1844).

The new operation $dx \wedge dy$ with symbol " $\wedge$ " is called "*wedge product* " or the *exterior product* . it appears in differential forms, we return to differential forms:

-in $\mathbb{R}$: 0- form is $f(x)$ and 1- form $df = f'(x)dx$
-in $\mathbb{R}^2$: 0-form is $f(x,y)$ and 1- form $df = Adx + Bdy$ and 2- form $df = Adxdy$.
-in $\mathbb{R}^3$: 0-form is $f(x,y,z)$ and $1-$form $dx + dy + dz$ and $2-$form is $dx \wedge dy + Bdx \wedge dz$ and the $3-$form is $Adx \wedge dy \wedge dz$.

We have seen the propriety $dxdy = -dydx$, it is called *skew*[14]*-symmetry*. In fact some proprieties:

Lemma:

Let α, β, γ be three forms and respectively then

- $\alpha \wedge \beta = (-1)^{lk}\alpha \wedge \beta \ \ lk$, which gives $\alpha \wedge \beta = -\beta \wedge \alpha$.
- $\alpha \wedge (\beta + \gamma) = \alpha \wedge \beta + \alpha \wedge \gamma$.
- $(\beta + \gamma) \wedge \alpha = \beta \wedge \gamma + \beta \wedge \alpha$.
- $\alpha \wedge (\beta \wedge) = \alpha \wedge (\beta \wedge)$, $\wedge$ is *associative*, unlike $x \times y$.
- $(c\alpha) \wedge \beta = \alpha \wedge (c\beta) = c(\alpha \ \wedge \beta)$ for a real.

To evaluate the product we need two vectors : $\mathbb{R}^2$ $v(a_1, a_2)$, $w(b_1, b_2)$ and the product is a determinant When $n = 2$ it is defined.

$$\alpha \wedge \beta = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

We plug vectors in . For product of two $2-$forms

$$(a_1 i + b_1 j) \wedge (a_2 i + b_2 j) = a_1 a_2 i \wedge i + a_1 b_2 i \wedge j + b_1 a_2 j \wedge i + b_1 b_2 j \wedge j$$

$$= a_1 b_2 - a_2 b_1$$

Generalize to $k-$forms . Which is a determinant. The gives the known vector product $a \times b$. $\wedge$ of two multicovectors is a generalization of $\times$. The exterior product of two 1-forms is the cross product.

[14] A matrix is *skew* if $A^T = -A$. Two lines in $\mathbb{R}^3$ are skew if they belong to two parallel planes.

we can define with a new structure and the operator ∧ . the structure is a vector space $\bigwedge^p V$ is the space of p –vector on V. What is $\bigwedge^0 V = \mathbb{R}$ is the real numbers and $\bigwedge^1 V$

$$\bigwedge^p$$

Area and volume:

The wedge product is for calculating surfaces and volumes. when $v \wedge v = 0$ (area $= 0$). $dv_2 \wedge dv_1 = -dv_1 \wedge dv_2$ (anti-symmetric, $a \wedge b$ is an oriented surface).

-The exterior product of a 1-form and a 2-form is the dot product.

-The exterior product of thee 1-forms is the triple product of vectors $v_1 \circ (v_2 \times v_3)$.

-The exterior product of two 2-forms is 0.

-The exterior product of two 1-forms is the cross product : find : $(Adx + Bdy + Cdz) \wedge (Ddx + Edy + Fdz)$. It is the vector(cross) product of two vectors.

Two differential forms $\omega = 2xydx \wedge dy - 3zdy \wedge dz$ so for $a(1,1,3)$ we have $\omega = 2dx \wedge dy - 9dy \wedge dz$ for two vectors $a(1,-1,3)$ and we have $a(2,1,1)$

$$\omega = 2xy \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} - 3z \begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix}$$

dx is 1 −form. as with $\wedge$ we calculate surfaces and volumes, the $dv_1 \wedge dv_2 \wedge dv_3$ is an oriented volume.

Exterior derivative:

the exterior derivative(***Cartan*** 1899) of a form ω of degree k. Every closed form is exact(in some cases)is every exact form is closed. The gradient(∇), the curl($\nabla \times$) and the divergence($\nabla \circ$) are examples of the exterior derivative. Let ω be a differential form then.

$$d\omega = \sum \frac{\partial \omega}{\partial x_i} dx_i \wedge dx_i$$

Examples:

Let $f(x) = xye^x$. If we apply the above formula to functions, the exterior derivative is simply the differential

$$df = (xye^x + ye^x)dx + xe^x dy$$

-let $= x^2yzdx + yzdx + (x + y)dx$. Then $df = d() \wedge dx + d() \wedge dy + d() \wedge dz$

Theorem:

the derivative of k-form is $(k-1)$-form.

a special cases of exterior derivative are the operators

$$\textit{function} \xrightarrow{\nabla} \textit{vector field} \xrightarrow{\nabla\times} \textit{vector field} \xrightarrow{\nabla\circ} \textit{function}$$

the *total* differential of f which is $\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$ is the exterior derivative of f. So df is just the gradient.

Theorem:

$-d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge \beta$
$-d(d\alpha) = 0$
$-d(f \wedge \omega) = df \wedge \omega$ if f is a function.
$-d(\alpha + \beta) = d\alpha + d\beta$

From $d(d\omega) = 0$ we have two important similar relations between differential operators,

$$\nabla \circ u = 0$$
$$\nabla \times u = 0$$

The propriety of exactness: if $d\alpha = 0$ then there exists $f : df = \alpha$. There is a similarity with differential operators

$\nabla \times u = 0$ then there exists v $: u = \nabla v$

$\nabla \circ u = 0$ then there exists v $: \nabla \times v = u$

Closed forms:

In the previous on differential equations, the condition $P_y(x,y) = Q_x(x,y)$ means that f is exact. we have said that a form α is closed if $\exists \beta : \alpha = d\beta$. Or $f(x,y) = c$ such that $df(x,y) = 0$

Definition

We say that a form ω is *closed* if $d\omega = 0$.

What relation is there between close and exact? We have the following lemma:

Lemma :

A differential form ω is exact then $d\omega = 0$.

We know that $(d\omega) = 0$, which means that every exact form is closed. The inverse is not always true . It is true under certain conditions

Poincaré lemma (1886)*:*

If the space is in $\mathbb{R}^n$ without holes and the C^1 then every closed form is exact.

Integration of forms:

as $\int_a^b f(x)dx = F(b) - F(a)$ which known as the fundamental theorem of calculus. return to exact to differential equation we know that the Green—Riemann theorem says that

$$\int_L Pdx + Qdy = \iint_D (P_x - Q_y)dxdy$$

$P_x - Q_y$ is the curl $\nabla \times$. So when we have the form is exact then $\int_L Pdx + Qdy = 0$. To calculate the line integral two methods:

-change of variables

-when the integrand is a total differential the is $\int_a^b df = f(b) - f(a)$.

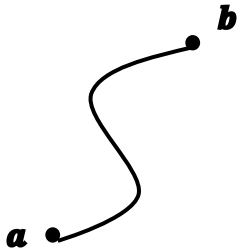


Fig. line integration.

It turns out that $\int_L Pdx + Qdy$ may depend on the path between two points in $\mathbb{R}^2$. Let two paths L_1 and L_2 between a and b. The value of $\int_{L_1} Pdx + Qdy$ may depends on path L_1. If we want that it does not depend , there is a condition on the function inside the integral the do not depend on the at if there exist a function such that $\nabla u = v$ which amounts to say that $\nabla \times u = 0$. For example $\int_{L_1} Pdr$ does not depend

◯ *Stokes theorem:*

One of the central result on the integration of differential forms is the Stokes theorem. A special case of the is theorem is the case $\int_a^b f(x)dx = F(b) - F(a)$..The theorem where we use differential to define the gradient, curl and divergence….

Another theorem due to Gauss is called the *divergence*[15] *theorem*

$$\int_V (\nabla \circ f)dv = \int_S fds$$

And above all Stokes theorem. We know from calculus that for a numerical function : $\int_a^b f(x)dx = F(b) - F(a)$ and .

$$\int_M d\omega = \int_{\partial M} \omega$$

[15] Also **Gauss** in 18 and **Green**.

Let ω be a differential form and $d\omega$ its exterior derivative.

Theorem

If we integrate ω on the boundary of a manifold M and $d\omega$ on M we find the same result.

We can write Maxwell equation become : $*F = 0$

$$\begin{cases} *F = 0 \\ *dF = 0 \end{cases}$$

The fundamental, the Green-Riemann theorem , divergence are special cases of Stokes theorem. The Hodge operators is defined as : k –form to $(k-n)$ −form

Lie derivative:

Let V and W vector fields. We want to evaluate the change of a field V along the flow of another field W. We define the Lie bracket(derivative or *commutator*)

$$[v, w] = vw - wv$$

noted sometimes $(\mathcal{L}_V W)_p$ and call it the *Lie derivative* of W with respect to V at the point p. We have :

$$(\mathcal{L}_V W)_p = -(\mathcal{L}_W V)_p$$

Examples

-let the form $\omega = xdx + ydy$ and the field

Exercise

1) integrate $\omega = -ydx + xdy$ on the path represented by a circle of radius $R = 1$.
1)Calculate the surface area of $y + z = -2x^2 - 1$ that lies in the rectangle $x = 1{,}2$ and $y = 0{,}3$.
1) show if the following forms are exact
* $xdx + ydy$
* $ydx + xdy$
1) simplify
* $5xdxdy + ydydx - dxdx + (x-1)dydy$
* $(ydx + xdy) \wedge (dx - ydy)$
1) Solve the following equations
* $y' = 1$.
* $y' - y = 1$.
* $y'' = 1$.
* $y' + xy = 1$.

1) Solve : $ay'' + by' + cy = d$.
1) Solve : $ay'' + by' = c$.
1) is the form $2xydx + (1 + x^2)dy = 0$ exact ? solve it.
1) what is a 0-form in $\mathbb{R}^n$? And a 1-form ?
1) show that : $\alpha \wedge \beta = -\beta \wedge \alpha$.
1) show that $\alpha \wedge \alpha = 0$ by two methods.
1) find : $(Adx + Bdy + Cdz) \wedge (Ddx + Edy + Fdz)$.
1) Show that $d\alpha \wedge d\beta \wedge d\alpha = 0$.
1) let $\lambda \in \mathbb{R}$, show that $\alpha \wedge (\lambda\alpha) = 0$.
1) find $d\omega$ when ω is the 0-form : xye^x.
1) find $d\omega$ when 1-form : $\omega = xydx + e^x dy$.
1) show that every exact form is closed. Apply on : $\omega = f_x dx + f_y dy$
1) show that $[v, v] = 0$.
1) define a tensor. A tensor has many directions.
1) Define a direct sum(or direct product) of v_1 and v_2.
1) Define a direct sum of V_1 and V_2
1) define a tensor product v_1 and v_2.
1) Define a tensor product of V_1 and V_2
1) Let$f(x, y) = x^3y + x$. Calculate Δf.
1) let$f(x, y, z) = yi + xj + xy^3k$. Calculate $\nabla \times f$.
1) let $f(x, y, z) = x^3 + y^2 - 3z^2$ find : ∇f, $\nabla \circ f$. Show that $\nabla \times \nabla f = \nabla \circ \nabla \times f = 0$.

Hyperspace

The Ancients knew many shapes like *polygons* which are plan figures in $\mathbb{R}^2$, others are in space which are called *solids* , and the interesting type of these are called *tetrahedra*. All these geometrical are situated in our physical space $\mathbb{R}^3$:

Polygons: plan figures bounded by line segments.
solids : space figures bounded by polygons.
Polyhedra : A polyhedron is usually a solid[16] with flat polygonal faces and solid angles.
Polytopes : A polytope is generalization of polygons to n dimensions. Polyhedra and polygons are

[16] It can be *hollow* : empty from inside.

examples of polytopes.

One of the main objectives of the Ancients was to measure areas of surfaces and volumes of solids. They were interested in regular shapes too. A regular polygon is one with equal sides and inside angles. A regular polyhedron is one with same regular polygons. They had discovered that there are 5 *regular* polyhadra only

cube, tetrahedron, octahedron, dodecahedron, icosahedron

Fig. Platonic solids.

called *Platonic solids*, a fact proved in the *Elements* of Euclid.

◯ *Spheres and Balls:*

Other very studied today like the general sphere S^n :

S^0 : the two point $\{a, b\}$.
S^1 : the circle(or 1 −sphere).
S^2 : the ordinary sphere called also 2 −sphere.
S^3 : the 3 −sphere, which is very important in topology. The bizarre thing is that since $S^2 \subset \mathbb{R}^3$ we can see S^2 and $\mathbb{R}^3$ but we cannot see $S^3 \subset \mathbb{R}^4$. How can we imagine S^3 and $\mathbb{R}^4$? We will see some strange phenomenon with the space $n = 4$. it seems that the space $n = 4$ is the border between real world and imaginary.
S^n : the n −sphere. A n −sphere is of the definition

$$S^n = \{x \in \mathbb{R}^{n+1} \ : \ |x| = 1\}$$

it is the *standard* sphere(some note it $\mathbb{S}^n$). So we remark that : $S^0 \subset \mathbb{R}^1$, $S^1 \subset \mathbb{R}^2$ and $S^2 \subset \mathbb{R}^3$ and in general $S^n \subset \mathbb{R}^{n+1}$. We can include a space of dimension k in $\mathbb{R}^n$ and $n - k$ is the *co-dimension* where $\mathbb{R}^0 = \mathbb{C}^0 = \{0\}$, and

$$\mathbb{R}^0 \subset \mathbb{R}^1 \subset \cdots \subset \mathbb{R}^n \quad , \mathbb{R}^\infty = \bigcup_n \mathbb{R}^n \text{ and } \mathbb{C}^n = \mathbb{R}^{2n}.$$

−Balls :

D^0 : point $\{a\}$
D^1 : interval $[a, b]$.
D^2 : disk it is the plane surface like the equation $\sqrt{x^2 + y^2} \leq 1$, it is bounded by a circle S^1.
D^3 : a ball, is one like the equation $\sqrt{x^2 + y^2 + z^2} \leq 1$, it is the volume bounded by a sphere S^2. If ∂ is the sign for boundary then $\partial D^3 = S^2$.
D^n : n-dimensional ball. A n-ball is of the definition

$$D^n = \{x \in \mathbb{R}^{n+1} \ : \ |x| \leq 1\}$$

We have in general $\partial D^n = S^{n-1}$ and $D^n \times D^m = D^{n+m}$

−torus : another very studied shape in topology:

T^2: $S^1 \times S^1$ is the torus , so unlike D^n: $S^1 \times S^1 \neq S^2$.

T^3 : $S^1 \times D^2$ is the solid torus.
T^n : $S^1 \times \ldots \times S^1$ is the n −torus. It is a subspace of $\mathbb{R}^{2n} = \mathbb{C}^n$.

A cylinder(or annulus) is $C = S^1 \times I$ where I is an interval.
We have $\partial S^1 = \emptyset$ and $\partial S^3 = \partial D^4 = T^3 \cup T^3$.

The precedent development leads us to the *product space*

Definition

Let X and Y two spaces. We define the topological space noted $X \times Y$ as the

Cartesian product(called also direct product)

next to product we use also :
− the *wedge* sum of two spaces: noted $X \vee Y$, it is a one point union of topological spaces, like for example the *rose* (bouquet of n circles)

$$S^2 \vee S^2 = S^2/S^1$$

Usually to understand a space X we consider a map between X and another space like the sphere S^2.
−there is also the *connected* sum of two spaces X and Y: sphere with torus give a torus.

–There is *smash* product of two spaces X and Y: which is noted $X \wedge Y$:

$$X \wedge Y = X \times Y / X \vee Y$$

◯ *Manifold:*

In his short career, German mathematician Bernhard **Riemann**$(1826 - 1866)$ authored two very important papers:

–*On the hypotheses which lie at the bases of geometry* (1854).
–*Theory of Abelian functions* (1857).

Both papers are implied in our subject. In the first paper he got the idea of n –dimensional spaces, the generalization of usual shapes to the space $n \geq 4$ and to consider it as usual as our space, he called it *Mannigfattigkeit* (multiplicity) which has been translated to the word *manifold*. **Poincaré** insisted in his paper of 1895

Nobody doubts nowadays that the geometry of n dimensions is a real object. Figures in hyperspace are as susceptible to precise definition as those in ordinary space, and even if we cannot represent them, we can still conceive of them and study them. So if the mechanics of more than three dimensions is to be condemned as lacking in object, the same cannot be said of hyper-geometry.

Every curve and smooth surface are two examples of manifold.

Fig. curves and surfaces are manifolds.

A space X is *embedded*(or imbedded) in Y as S^1 is in $\mathbb{R}^3$. The difference between intrinsic and extrinsic if we twist a band in $\mathbb{R}^3$, it does not change intrinsically but the way it is embedded is changed extrinsically.

–0-manifold : a point or a finite number of points.

−1-manifold : a curve in the plan : $x(t)i + y(t)j$. Or $y = f(x)$ Also a circle S^1, an annulus $S^1 \times [0,1]$.

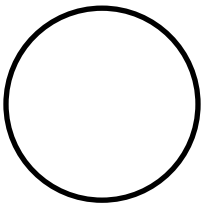


Fig. a smooth 1 −sphere

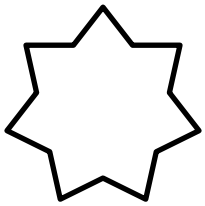


Fig. A topological 1 −sphere

−2-manifold : are surfaces in $\mathbb{R}^3$. a 2-manifold can be of the equation $F(x,y,z) = 0$ or $z = f(x,y)$. An example is the sphere S^2. The ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ which is a bounded surface and the hyperboloid $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ which is unbounded. Other manifolds are the torus T^2, the Mobius band[17] and the Klein bottle.

−3-manifold. Like the 3 −sphere which is the boundary of a ball : $S^3 = \partial D^4$. It is difficult to visualize and imagine this manifold, it is as impossible with the graph of a function in $\mathbb{C}$. It can be presented by the equation $|z_1| + |z_2| = 1$. Or with quaternions by the equation $|q| = 1$. It has been imagined by German Heinz **Hopf**$(1894 - 1971)$ using his *Hopf fibration*. **Hopf** discovered that there is a map from S^3 to S^2 :

$$H : (y_1, y_2, y_3, y_4) \longrightarrow (x_1, x_2, x_3)$$

such that :

$$\begin{cases} x_1 = 2(y_1 y_2 + y_3 y_4) \\ x_2 = 2(y_1 y_4 - y_2 y_3) \\ x_3 = ({y_1}^2 + {y_3}^2) - ({y_2}^2 + {y_4}^2) \end{cases}$$

which is not null-homotopic, we must verify that on a sphere : ${x_1}^2 + {x_2}^2 + {x_3}^2 = 1$.

$-n$ −maniofld : can be defined by a system of equation as did **Poincaré**(1894)

[17] Discovered by German August **Mobius**$(1790 - 1868)$ in 1858 and independently by Johan **Listing**$(1808 - 1882)$ in the same year.

$$\begin{cases} f_1(x_1, \dots, x_n) = 0 \\ \dots \\ f_n(x_1, \dots, x_n) = 0 \\ h_1(x_1, \dots, x_n) > 0 \\ \dots \\ h_m(x_1, \dots, x_n) > 0 \end{cases}$$

This is a manifold is called *algebraic variety*[18]. The manifold can be defined as $M = f^{-1}(c)$ where $c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ the manifold can be linear and

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = c_1 \\ \dots \\ a_{k1}x_1 + \cdots + a_{kn}x_n = c_k \end{cases}$$

$\mathbb{R}^n$ is a linear manifold . **Riemann** (1854) generalized Gauss' work to higher dimensions, this space is known to be called *hyperspace*. A *hyperplane* is the space that separates n −dimensional space into two regions. A line and plane are hyperplane for 2 and 3 dimensions. The idea is to generalize the idea to $n > 3$ so hyperplanes are the $(n-1)$ −flats.

◯ *Proprieties of Manifold:*

Manifolds can be very complicated. There are some proprieties around them:

−*Orientation*: a manifold can be either oriented or not. The orientation is about the sense(direction). two vectors $v_1 = \alpha v_1 = v_2$ has the same direction if $\alpha \geq 0$. Vectors v_1, v_2 such that $v_1 = Av_2$ have the same orientation if the $det(A) > 0$. Möbius band is not oriented. Oriented manifolds were introduced by **Klein** for surfaces and generalized by von **Dyck** for manifolds of arbitrary dimension.
−*Finiteness* : a manifold can have a finite or infinite dimension.

−*Boundness* : can also be either bounded or unbounded. The ball , disk and solid torus are finite . $x^2 + y^2 - z^2 = 1$ the is not finite.. $\mathbb{R}^n$ is not bounded ..if a manifold is finite but it is not necessary with a *boundary*. The ball , disk and solid torus have no boundaries but the sphere, circle and torus have no boundaries. a manifold with boundary : we have

$$\partial(X \times Y) = (\partial X \times Y) \cup (X \times \partial Y)$$

If the boundary of X is empty $(\partial X = \emptyset)$ then X is closed.

[18] **Poincaré** named manifold by the French word *varieté,* then in the subsequent current , variety in English has become a special manifold in algebraic geometry.

Space	boundary ∂
$[a,b]$	$\{a,b\}$
S^1	$\emptyset$
S^2	$\emptyset$
T^2	$\emptyset$
D^2	S^1
D^3	S^2
Mobius band : $I/\sim$	S^1
cylinder	

Fig. Boundaries of some spaces.

We have $\partial([a,b]) = \{a,b\}$ and $\partial(S^1) = \emptyset$. All S^1, T^2 and S^2 have no boundary.

−*Closeness* : also called openness if a manifold is bounded and closed it is compact.

If the boundary of a compact manifold M is empty ($\partial M = \emptyset$) then M is closed.
−*Connected or disconnected* : **Poincaré** took as a manifold the curve $y^2 + x^4 - 4x^2 + 1 = 0$ which is not connected but into two connected manifolds : $y^2 + x^4 - 4x^2 + 1 = 0$ for $x > 0$ and $y^2 + x^4 - 4x^2 + 1 = 0$ for $x < 0$. Another not connected plan manifold is *elliptic* curves form $y^2 = x^3 + ax + b$.

Betti to deal with connectivity in a general way :how many pieces can we remove in a manifold without disconnect it ? he introduced boundary(the sphere and torus have no boundary). **Mobius** made the earliest efforts to classify manifolds, he his famous *ruban* with one side . **Poincaré** named manifold *varieté* which becomes *veriety* in English. A *veriety* is a special manifold. It is used especially in algebraic geometry.

Klein bottle has neither interior nor exterior. It is not the surface of any solid. The same is true for the projective plane.

◯ *Topological Manifold:*

Now we are ready to give a formal mathematical definition of a manifold. A manifold to mathematician is a figure(curve, surface,…) that should be *well behaved* to be studied. So the idea is to be a space(line, surface,…) that looks locally like Euclidian space $\mathbb{R}^n$.

Definition

a manifold is a Hausdaurff second countable space that is homeomorphic to the Euclidian space.

This definition of German mathematician Herman **Weyl**$(1885-1955)$ in his book *Riemann surface*(1913).

Let M a Hausdaurff second countable space. A *chart* (U,φ) is $U \subset M$ with a map φ such that $\varphi : U \longrightarrow$
It is called *coordinates* because the image is coordinates in $\mathbb{R}^n$. Is there be many of them (U_i, φ_i)

Fig. Chart

a compatible chart (φ, U).
a transition map $\phi \circ \varphi^{-1}$.

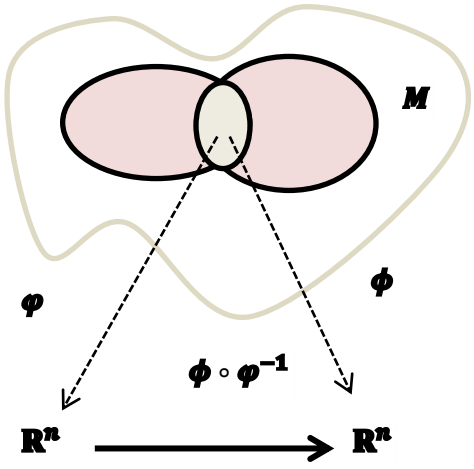


Fig. Charts

When the charts cover the manifold

their union $\cup \quad U_i$ is called an *atlas*[19] $\mathcal{A} = \cup \quad U_i$.

The atlas defines the topological structure of the manifold. When the *transition* map $\phi \circ \varphi^{-1}$ between local coordinate charts is continuous we say that M is topological manifold. We the homeomorphism by charts ,

A *maximal* atlas is one that contains all atlases .

set of *compatible* charts that cover a manifold. When M is abstract which we can do calculus. So is immersion o submersion. A chart is a coordinate , a *compatible* chart

The chart may be not unique. A manifold of dimension n is noted M^n can be .

A cross is not a manifold because ta the point of intersection

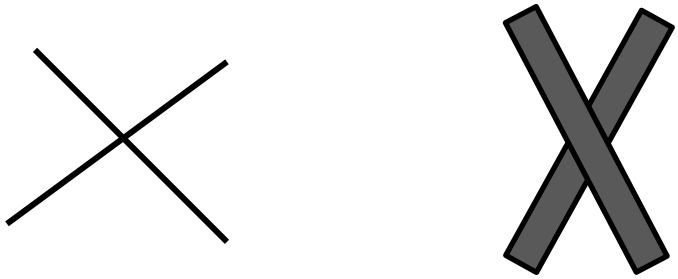


Fig. cross is not but the right is.

A topological manifold is connected iff is path connected.

Examples:

- $\mathbb{R}^n$ is a topological manifold.
- a line.
- a circle S^1.
- a torus $S^1 \times S^1$.
- for example a sphere, which is a manifold of dimension 2. By the stereoprojection to $\mathbb{R}^2$ we now an abstract example of manifold
-let the $GL_n(\mathbb{R})$ of matrices is a manifold.
-the set $SO(n)$.of special orthogonal matrices is a manifold of dimension $\frac{n(n-1)}{2}$.

[19] An atlas was introduced in geography by Flemish Gerardus **Mercator**$(1512 - 1594)$ and is the collection of maps covering a part of Earth. Atlas is the name of a pagans' god in Greek mythology.

– *Piecewise linear manifold*: noted PL is a manifold whose transition function is piecewise linear.

To understand the topology of a manifold we use Morse function.

Exercise

1)what is the area of a square of side $r = \sqrt{2}\ cm.$ the volume of a cube of side $r = \sqrt{3}\ cm.$

1) tell if the vectors v and u linked by the relation $\begin{cases} v_1 = 2u_1 + u_2 - u_3 \\ v_2 = -u_1 + 3u_1 + u_1 \\ v_3 = u_1 - u_1 - 5u_1 \end{cases}$ have the same orientation in space.

1) Is $[0,1]$ a manifold ?

1) Is $]0,1[$ a manifold ? what is $\partial(]0,1[)$?

1) what is $\emptyset \times X$?

1) what is $S^1 \times D^1$?

1) what is $S^1 \times S^1 \times S^1$?

1) what is $S^1 \times S^1 \times [a,b]$? $a, b \geq 0$

1) is there a homeomorphism between S^m and S^n when $n \neq m$?

1) find $\partial(T^2)$.

1) find $\partial(D^2 \cup [a,b])$.

1) find $\partial(S^3)$.

1) Show that $int(M) \cap \partial M = \emptyset$.

Groups

In He what now the *Poincare group* or *fundamental group*. The concept of *group* is already in 18 of algebraic equations. And a set G is a group with an internal law $*$ if :

-$\forall g_1, g_2, g_3 \in G\ : g_1 * (g_2 * g_3) = (g_1 * g_2) * g_3.$

-$\exists e \in G\ , \forall g \in G : g * e = e * g.$

-$\forall g \in G$, $\exists\, g^{-1} : g * g^{-1} = g^{-1} * g = e$.

The first axiom that $*$ is *internal* or G is *closed* for $*$. The group is also noted $(G,*)$. In what follows we write $g_1 g_2$ in place $g_1 * g_2$ for simplicity.

Remarks:

−if the number of elements of G is finite we say that G is of finite *order* and we note this number by $\sharp\, G$, if not it said group of infinite order.

− If $g_1 g_2 = g_2 g_1$ for all $g_1, g_2 \in G$, then G is we say that the group G is *Abelian*.
−we say that an element $g \in$ G is of *order k* if there exists a least naturel integer k such that $g^k = e$ ■

The reader can verify that the followings sets are infinite order groups[20] (groups of infinite order):
$(\mathbb{Z},+)$, $(\mathbb{Q},+)$, $(\mathbb{Q}^*,\bullet)$, $(\mathbb{R},+)$, $(\mathbb{R}^*,\bullet)$, $(\mathbb{C},+)$, $(\mathbb{C}^*,\bullet)$.

The group $(\frac{\mathbb{Z}}{n\mathbb{Z}},+)$ is an additive abelian group of finite order, and $\sharp\, (\mathbb{Z}/n\mathbb{Z}\,,+) = n$. The group $\left(\frac{\mathbb{Z}}{n\mathbb{Z}},+\right) = (\mathbb{Z}_2,+) = \{0,1\}$ is an additive abelian group of finite order, $\sharp\, (\mathbb{Z}_2,+) = 2$.

◯ *Subgroup:*

Let G a group, a set $H \subset G$ is a subgroup if

$-\ \forall\, h_1\,, h_2 \in H$ we have $h_1 h_2 \in H$.
$-\ \forall\, h \in H$ we have $h^{-1} \in H$.

If H is a subgroup of G we note $H \preccurlyeq G$. We have examples of subgroups $(\mathbb{Z},+) \preccurlyeq (\mathbb{Q},+) \preccurlyeq (\mathbb{R},+) \preccurlyeq (\mathbb{C},+)$, $(\mathbb{Q}^*,\bullet) \preccurlyeq (\mathbb{R}^*,\bullet) \preccurlyeq (\mathbb{C}^*,\bullet)$. The subgroups $(\{e\},*)$ and $(G,*)$ are trivial subgroups.

Some important : applications

Lagrange theorem:

Let G finite group and H a sub-group of G , then the order of H divides the order of G :

$$\frac{\sharp G}{\sharp H} \in \mathbb{N}.$$

The number $\frac{\sharp G}{\sharp H}$ is called the *indice of H in G* and noted $[G \div H]$.

[20] And also $(\mathbb{Z}^n,+)$, $(\mathbb{Q}^n,+)$, $(\mathbb{R}^n,+)$, $(\mathbb{C}^n,+)$.

◯ *Klein program*

The idea of using group in geometry is very original and was first by German mathematician Felix **Klein**$(1849-1925)$ in 1872. The concept of group is intimately linked to *symmetry*. **Klein** had to a lecture at the university of Erlangen(of German) and he proposed a program that to classify geometry by the idea of *transformation groups*

The geometry is characterized by the group of transformations that preserve elementary proprieties

This philosophy was following the of geometry in the XIX century and invention of hyperbolic and parabolic, projective,… geometries. On algebraic topology (which he called Analys Situs) he said:

in the so-called analysis situs we try to find what remains unchanged under transformations resulting from a combination of if infinitesimal distortions. Here, again we must distinguish whether the whole region, all space, for instance, is to be subjected to the transformation, or only manifoldness contained in the same, a surface. It is the transformation of the first kind on which we could found space geometry. Their group would be entirely different in constitution from the groups heretofore considered. Embracing as it does all transformations compounded from(real) infinitesimal point-transformations, it necessary involves the limitation to real space-elements, and belongs to the domain of arbitrary functions. This group of transformations can be extended to advantage by combining in with those real collineations which at the same time effect the region at infinity.

transformations or their combinations:

-translation
-rotation
-reflection

These transformations can preserve or not the proprieties:

-parallelism //
-length
-colinearity
-orthogonality $\perp$
-cross ratio of 4 colinear points.
-congruence of figures
-area

the three related geometries are

Affine: according to Mobius transforms lines to lines, parallels to parallels, rations…the affine group G_A is a group of invertible affine transformations.

Euclidian : transformation that preserve isometries. They are the proprieties that do not change by motion. The Euclidean group G_E is a group

Projective: transformation that preserve the cross ratio, projective transformation that takes a conic to itself lead to non-Euclidean hyperbolic geometry.. The Projective group G_P is a group

We have

$$G_E \subset G_A \subset G_P$$

And

$$\mathbb{E}^n \subset \mathbb{A}^n \subset \mathbb{P}^n$$

Another mathematician who did well in this domain was Norwegian Sophus **Lie**$(1842 - 1899)$ his continuous groups known as *Lie groups*. In the same manner for a topologist as a doughnut is equivalent to a cup,

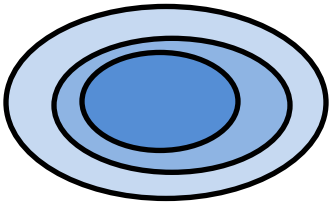


Fig. three geometries

◯ *Special groups:*

Between two groups are the groups of matrices , a matrix is a linear transformation . Three interesting are reserving some interesting proprieties

− *The linear group* : LG is of invertible square matrices , if n is the order of the matrices it is noted $LG(n, \mathbb{R})$, the law compositon is the matrix multiplication.
− *The group of orthogonal* : O: matrices, an orthogonal A matrix is such that $A^{-1} = A^T$, we know that $det(A) = det(A^T)$. An orthogonal transformation preserves the inner product of vectors.
−*The special orthogonal group* (SO): matrices an orthogonal matrix with $det(A) = 1$. The group of rotations

we can have maps

$$SO(n) \preccurlyeq O(n) \preccurlyeq GL(n)$$

$GL(3,\mathbb{R})$, $O(3)$ and $SO(3)$ are called *topological groups*.

The circle for the elements are complex numbers $z = e^{i\theta}$ is a group and the multiplication law • called the *circle group* noted $\mathbb{C}^{\times}$

$e^{i\theta_1} \bullet (e^{i\theta_2} \bullet e^{i\theta_3}) = (e^{i\theta} \bullet e^{i\theta}) \bullet e^{i\theta}$
$e^{i0} \bullet e^{i\theta} = e^{i\theta}$
$e^{i\theta} \bullet e^{-i\theta} = e^{i0} = 1$

It is also abelian as $z_1 \bullet z_2 = e^{i\theta_1} \bullet e^{i\theta_1} = e^{i(\theta_1+\theta_2)} = e^{i(\theta_1+\theta_2)} = z_2 \bullet z_1$. Every finite abelian group can decomposed in.

◯ *Normal group*

A subgroup is normal if and we note $G \lhd H$. A group G is *simple* if the only normal groups are $\{e\}$ and G. If then we can study H in place of G.

◯ *Quotient group:*

$\frac{\mathbb{Z}}{n\mathbb{Z}}$ is the quotient group of $\mathbb{Z}_n$.

◯ *Homomorphisms of Groups:*

Between two groups we can have maps f from E to F that are

-A homomorphism : called also morphism(a general term)
-Isomorphism : f is a homomorphism and it is bijective and.
-automorphism : it is an isomorphism the bijection from E in E.

Linearity is a morphism between two vector spaces which are groups $(V,+)$

$$f : (\mathbb{R},+) \longrightarrow (\mathbb{R},+)$$
$$f\,(x+y) \longrightarrow f(x) + f(y)$$

V here is $\mathbb{R}$. $Ln(x)$ is a homomorphism of groups$(\mathbb{R},+)$, e^x homomorphism of, the determinant $det(A)$ is homomorphism of groups $GL(\,)$.

An interesting is that two groups can *isomorphic*. A morphism group is $\phi : (G, e_1) \longrightarrow (G, e_2)$. Such that $\phi(g_1 g_2) = \phi(g_1)\phi(g_2)$. For every homomorphism of groups we can define two sets :

– the kernel : noted ker and defined as

$$Ker(\phi) = \{g \in G_1 : \phi(g) = 1_G\}$$

It measures the injectivity of ϕ. We can prove that is a subgroup of E.

– the image : not ker and defined as

$$Im(\phi) = \{g \in G_2 : \phi(g) = g\}$$

If two finite groups G_1 and G_2 are isomorphic then $|G_1| = |G_2|$.

Homotopic group

group is A path

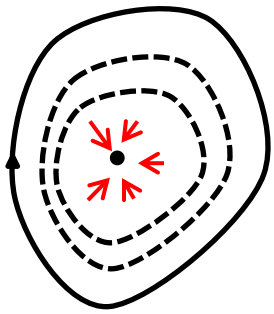


Fig. to a point.

A covering space: they help in computing homotopy groups. We look if X is a cover space of Y or not.

Exercises

1) Define $\emptyset$ with two methods.
1) is $\emptyset$ a group?
1) show that e is unique .
1) let $f(x) = x - 1$ and $g(x) = x^2 + 1$. Find fog and gof.
1) Are $(\mathbb{N}, +)$, $(\mathbb{N},\bullet)$, $(\mathbb{Z},\bullet)$, $(\mathbb{Q},\bullet)$, $(\mathbb{R},\bullet)$, $(\mathbb{C},\bullet)$ groups?
1) why are $(\mathbb{Z}^{*n},\bullet)$, $(\mathbb{Q}^{*n},\bullet)$, $(\mathbb{R}^{*n},\bullet)$, $(\mathbb{C}^{*n},\bullet)$ not groups ?
1) show that the operation in the circle group is internal.
1) Show that $(\{e\},*)$ is a group.
1) Show that g^{-1} is unique.
1) let $H \leqslant G$. Show that $e \in H$.

1) let H_i subgroups of G. Show that $\bigcap H_i$ is a subgroup of G.
1) determine all subgroups of group of prime order p.
1) are the following matrices orthogonal ?

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

1) let A be an orthogonal matrix. Show that $det(A) = \pm 1$.
1) Show that the composition law in $O(n)$ is internal.
1) Let ϕ a homomorphism of groups G_1 and G_2 , show that : $\phi(e_1) = e_2$
1) show that : $\phi(g^{-1}) = \big(\phi(g)\big)^{-1}$.
1) Let f linear, show that $f(0) = 0$.
1) Show that S_3 is not abelian.
1) Show that $\equiv$ in $\mathbb{Z}_n$ is an equivalence relation. There is a canonical projection $\pi : \mathbb{Z}_n \longrightarrow \mathbb{Z}_n/\equiv$
1) Show that the det : $GL(\mathbb{R}) \longrightarrow \mathbb{R}^*$ is not an isomorphism.

The Fundamental Group

We have seen that we can paths in classes of equivalence. We can now the same with loops and discover that the classes can form a group structure. The operation between the paths γ is *composition* and noted 'o'. concatenation of two paths or more, in this case : $\gamma_1(1) = \gamma_2(0)$.

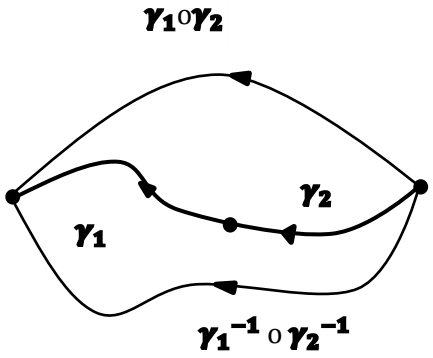


Fig. product of two paths.

In the previous we remark $\gamma_1(1) = \gamma_2(0)$. We will see that the fundamental group is a set of special path equivalences γ that verifies the axioms of a group. The difference between general paths and

loops.

◯ *Composition of loops*

the special cases are loops. A loop is based at the point x_0. The point x_0 is arbitrary. So all loops based at a point constitute a homotopy classes. If $\gamma(0) = \gamma(1) = x_0$ then is based at x_0

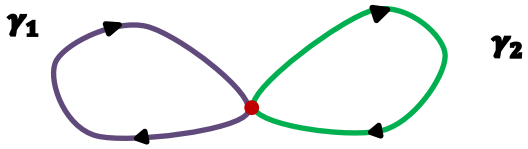


Fig. the wedge sum of two loops.

We that $\gamma_1\gamma_2 \neq \gamma_2\gamma_1$ And ${\gamma_1}^2{\gamma_2}^{-3}$? It is difficult to how loops compose. And sure that this operation is well defined.

Theorem

Let X a topological space. The set of equivalence classes $[\gamma_i]$ of loops at base point x_0 in X is a group . it is noted $\pi_1(X, x_0)$.

$- \ [\gamma_1]\mathrm{o}[\gamma_2]$
$- \ ([\gamma_1]\mathrm{o}[\gamma_2])\mathrm{o}[\gamma_3] = \ [\gamma_1]\mathrm{o}([\gamma_2]\mathrm{o}[\gamma_3])$
$-\exists[\gamma_1]\colon [\gamma_1]\mathrm{o}[\gamma_2] = [\gamma_1]$

the elements of $\pi_1(X, x_0)$ are the homotopy equivalence classes. Poincare is about groups:

I needed it for the study of non-uniform functions of two variables. I needed it for the study of the periods of multiple integrals and for applying this study to the development of the perturbing function. Finally, I glimpsed in Analysis Situs a way to approach an important problem in group theory: the search for discrete groups or finite groups contained in a given continuous group.

So this group noted $\pi_1(X, x_0)$ and invented by **Poincaré** is called a *first homotopy group* or *fundamental group*. Sometimes called *Poincaré group*

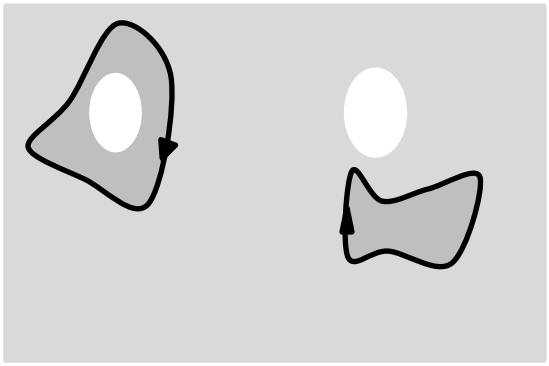


Fig. Different loops

A fundamental group of E is seen as $f: S^1 \longrightarrow X$ and in general $\pi_n(E)$ $f: S^n \longrightarrow E$. when $\pi_1(E) = \{e\}$, the fundamental group is ***trivial*** if it contains the identity element e. If the fundamental groups of two spaces are isomorphic they are homeomorphic. We often ***calculate*** the group.

Theorem

The fundamental group of the circle is isomorphism to $\mathbb{Z}$.

$$\pi_1(S^1) \approx \mathbb{Z}$$

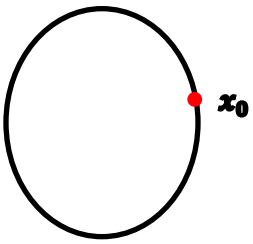


fig.

or we can simply write $\pi_1(S^1) = \mathbb{Z}$. It is an infinite cyclic group. We have also $\pi_1(\mathbb{R}) = \mathbb{Z}$.

X	$\pi_0(X)$	$\pi_1(X)$	$\pi_2(X)$	$\pi_3(X)$	$\pi_4(X)$
S^1		$\mathbb{Z}$	$\{e\}$	$\{e\}$	
S^2		$\{e\}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2$
S^3		$\{e\}$	$\{e\}$	$\mathbb{Z}$	$\mathbb{Z}/2$
D^2		$\{e\}$			
$SO(n)$					

Fig. some homotopy groups of usual manifolds.

Proposition

Let X a topological space. If $n < m$ then $\pi_n(S^m) = \{e\}$.

So we have the result

Two spaces are homeomorphic $\Rightarrow$ have the same homotopie type

Two spaces have the same homotopie type $\not\Rightarrow$ they are homeomorphic

$\pi_1(X)$ is an invariant that does not depend on the homotopy type. If $\pi_1(X)$ is not trivial, there is a possible hole in X.

Some results like

Theorem

Let X a topological space. If $\pi_1(X) = \{e\}$ iff X is simply-connected.

If X is path connected , the group $\pi_1(X)$ is independent of the base point. If $\pi_1(X) = \{e\}$ then every loop in X is contractile. We know $\pi_1(S^0) = \mathbb{Z}$ how to calculate it?

It is very difficult generally to calculate homotopic groups(which can be cyclic, free, abelian, non abelian,…) The fundamental group of **8** figure is a free group . There are higher order homotopy groups $\pi_n(X)$. The group $\pi_1(X)$ can be non-commutative, but :

Theorem

Let X a topological space. For $n > 1$ all homotopy groups $\pi_n(X)$ are abelian.

Seifert-Kampen theorem:

In 1930, two mathematicians Herbert **Seifert**$(1907 - 1996)$ and Dutch Egbert van **Kampen**$(1908 - 1942)$ proved a new theorem on the fundamental group.

Seifert-Kampen Theorem

Let X and Y two topological spaces. The fundamental group of $X \times Y$ is given by : $\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y)$.

Where $\times$ is the product. An example is circle and the torus $\pi_1(T^2) = \pi_1(S^1 \times S^1) = \pi_1(S^1) \times \pi_1(S^1) = \mathbb{Z}^2$. This theorem makes it easier to calculate $\pi_n(X)$ in some cases.

X	$\pi_0(X)$	$\pi_1(X)$	$\pi_2(X)$	$\pi_3(X)$	$\pi_4(X)$
T^2		$\mathbb{Z}^2$	$\{e\}$		
$S^1 \times [0,1]$		$\mathbb{Z}$			
$\mathbb{R}^n$		$\{e\}$			

Fig. some homotopy groups of product manifolds.

We remark that for $n \geq 2 : \pi_n(S^n) = \{e\}$. for wedge sum: : $\pi_1(X \vee Y) = : \pi_1(X) * \pi_1(Y)$ where the free product of groups. $\pi_1(\mathbb{R}^n) = \{e\}$ $\forall n = 1,2, \ldots . : \mathbb{R}^n$ is contractible and simply connected. For the plan any closed curve on $\mathbb{R}^2$ can be continuously deformed to an identity path.

◯ *Isomorphic Groups*

The *isomorphism problem* is central in group theory and algebraic topology. We say that two groups p isomorphic if they

Theorem ????

Let X and Y be two topological spaces. If X and Y are homeomorphic then $\pi_1(X), \pi_1(Y)$ are isomorphic.

So for the groups $\pi_1\left(S^1\right) = \{e\}$ and $\pi_1(\mathbf{8}) = \{a, b\}$, there is no hemeomorphic between S^1 and the $\mathbf{8}$ figure ($\mathbf{8} \cong S^1 \vee S^1$)because the groups are not isomorphic.

And The inverse is not true in general.

Theorem ????

Let X and Y be two topological spaces. If $\pi_1(X) = \pi_1(Y)$ then X and Y are isomorphic.

In 1933 , German Heinz **Hopf**$(1894 - 1971)$ discovered that the group $\pi_3(S^2)$ is isomorphic to $\mathbb{Z}$.

◯ *The Poincaré Conjecture:*

The Poincaré conjecture[21] says

A compact connected 3 *−manifold is topologically a* 3 *−sphere if it is simply connected.*

In other

$$\pi_1(X) = 0 \Longrightarrow X \cong S^3$$

If $n = 1,2$, it says : the only closed surface that is simply connected is the sphere S^2. The conjecture can be stated for all n.so In the general case

$$\pi_1(X) = 0 \Longrightarrow X \cong S^n$$

A lot of person tried either to prove it or disprove it like British John **Whitehead**$(1904 - 1960)$, American Hassler **Whitney**$(1907 - 1989)$, Greek Christos **Papakyriakopoulos**$(1914 - 1976)$ …what is sure is that the conjecture has much in the development of topology and its technics in XX century.

Surprisingly the conjecture when $n \geq 5$ was proved by American Stephen **Smale**$(193 -)$ in 1961 for smooth manifolds, he used his theory of *h-cobordism*[22]. Afterwards, John **Stallings**$(1935 - 2008)$ and

[21] **Poincaré** his conjecture : "*The* 3 *−sphere is the only closed 3-dimetsional manifold with trivial Betti and torsion number*" which is a wrong assertion. A counterexample is his *Poincaré sphere*(1904). It is the coset space $SO(3)/I$, where I is the group of rotational symmetries of icosahedron of order 60.

[22] Cobordism theory was invented by French René **Thom**$(1923 - 2002)$ in his PhD thesis (1951).

Thurston

Christopher **Zeeman**$(1925-2016)$proved it for PL manifolds. In Maxwel **Newman**$(1897-1984)$ and E. H. **Connell** proved it for general topological manifolds. The case $n=4$ was more difficult and proved by American Michael **Freedman** $(1951-)$ in 1982. The ideas of topology to developed and relations in the term of analysis and geometry, American William **Thurston**$(1946-2012)$ developed new ideas in the 1970's, he that a lot of 3-manifolds are hyperbolic, so on to 3 dimensions spaces and his *geometrization conjecture*(1982) which implied the Poincaré conjecture. It says:

$-$in 2 dimensions: geometries: plane, hyperbolic and spherical. Classification is easy under the *uniformization theorem* proved by Poincaré in 1907.

$-$in 3 dimensions : there are eight geometries: plane, hyperbolic ,spherical, Nil, Sol. 3-minofold does not always admit a structure.

So the idea that :

$$\textit{Geometrization conjecture} \implies \textit{elliptization conjecture} \implies \textit{Poincaré conjecture}$$

Thurston proposed a geometric description of 3-manifold in which Poincaré conjecture is a special case. He proved his conjecture for some cases like the Hacken manifolds. At the same year American Richard **Hamilton**$(1944-2024)$ introduced *Ricci flow* noted $Ric(t)$ which is the solutions of the equation

$$\frac{\partial g(t)}{\partial t} = -Ric(g(t))$$

Where g is the Riemannian metric. It looks evolution equations like the heat equation $\frac{\partial u(x,t)}{\partial t} = \alpha^2 u(x,t)$. A link between geometry and partial differential equations(PDE). **Hamilton** intended a program to attack the Poincaré conjecture with some success but difficulties emerged because of the formation of singularities. Missing details were completed by Russian **Perelman**$(1966-)$ in $2002-2003$ who, used a technics **Cheeger** and called *surgery*[23]. So this that all $3-$manifolds can be using 8 geometries only.

[23] Surgery originated in the work of Russian topologist **Pontriagin**(1-19) then developed by **Milnor**() in 1961.

Exercices

1) Show that the relation is an equivalence relation.
1) Find : $\pi_1(T^n)$.
1) find : $\pi_1(S^1 \times D^2)$.
1) what is $\pi_1(S^1 \vee S^1)$?
1)Show that the heat equation $u_t = \alpha^2 u_{xx}$ is a linear parabolic PDE. Solve it by the Fourier method.
1)Show that the following are equivalence relations.
* $=$ in $\mathbb{C}$
* $\parallel$ in the plan $\mathbb{R}^2$
* $\equiv$ in $\mathbb{Z}_n$

1)Show that the following functions are harmonic.:
* $u(x,y) = xy$
* $u(x,y) = ax - ay$.
* the real and imaginary parts of analytic function.

1)Show that $f\ (x) = \frac{1}{x(x-1)}$ is a bijection between $\mathbb{R}$ and $]0,1[$. Is it a homeomorphism?
1) What is $\pi_0(I) =$?

Classification of Manifolds

The main problem of topology of manifolds is the characterization of manifolds by means of algebraic invariants. Also the *homeomorphism problem* if two manifolds are homeomorphic. Linked to it is the *isomorphism problem* of groups. In 1892 , **Poincaré** tried to classify manifolds of dimension greater than 2. He on Riemann work

The classification of algebraic curves into types rests, according to Riemann, on the classification of real closed surfaces, based on the point of view of Analysis Situs. An immediate induction shows us that the classification of algebraic surfaces and the theory of their birational transformations are intimately linked with the classification of real closed surfaces in the space of five dimensions based on the point of view of Analysis Situs.

In 1919, American James **Alexander** $(1888 - 1971)$ showed that homotopy and homology groups are not sufficient to characterize 3 −manifolds. So we mention some proprieties on manifolds:

-finiteness, closure: compact without boundary, some are compact with boundary.
-number of holes or genus g. Genus of the cylinder is 0…

-orientation, A Mobius strip is not oriented(one side)…
-Euler characteristic that has a relation with g through the relation : $\chi(E) = 2 - 2g$
-order of connectivity presented by Betti numbers.
-for $n \leq 3$ topological, smooth and PL manifolds are the same.

◯*1-dimensional manifolds*

The classification of orientable closed connected curves is by the theorem:

Theorem

Every smooth connected 1 −dimensional manifold is diffeomorphic either to S^1 or to a real interval.

The proof[24] is . S^1 is the only compact connected 1 −manifold up to a homeomorphism.

◯*2-dimensional manifolds*

The orientable closed connected 2 −dimensional manifolds are completely characterized by their genus g, which determines their diffeomorphism type in a unique way.

Theorem

Every closed oriented surface is a sphere or a torus.

or

Theorem:

Any compact 2-manifold is the connected sum $\#$ of spheres or tori.

The fundamental group is sufficient to classify the closed oriented $2-$ dimensional surfaces but it is insufficient in higher dimensions. We can add:
-handle.
-crosshandle.
-crosscap.
-perforation.

[24] See **Milnor** : *Topology, from a Differential Point of View* p. 55.

So close surfaces (torus, ellipsoid,…) are topologically equivalent to the sphere with handles or crosscaps. In a similar fashion *non-orientable* two-dimensional manifolds are classified and were well understood in the XIX century. Italian **Bianchi**().

It was German August **Mobius**$(1790-1868)$ in 1863 who finished the classification of closed topological surfaces, every surface is homeomorphic to a standard surface with g holes. Then came the classification of non orientable surfaces by German mathematician Walter **Dyck**$(1856-1934)$ in 1888. Another proof was given by Dehn() and Haggard() in 1907. A method to do this is given by American mathematician Hassler **Whitney**$(1907-1989)$ called *Whitney trick* , its essence is to remove the intersection points between two manifolds. Another proof was found in 1992 by American John **Conway**$(19-20)$ called *zero irrelevancy* or *zip proof*.

◯ *3-dimensional manifolds*

The most important invariant for 3-manifolds is the fundamental group. In 1882, German Felix **Klein**$(1849-1925)$ described a new composed of two Mobius strips and a new surface was invented called the *Klein bottle* , it is a 3-manifold without volume. For 3-manifolds, there are no essential differences between smooth, piecewise a, and topological manifolds. Each $3-$manifold has a unique smooth structure.

When we make a transition from 2-dimensional to 3-dimensional manifolds difficulties grow astronomically, and up to the present day there is no classification of 3-dimensional manifolds. **Poinacré** in his 5th supplement studied 3-diemtional manifolds. In 1910 German Max **Dehn**$(1878-1952)$ found infinitely many homology spheres. In 1935, **Whitehead** found an open 3-dimestional manifold that is a simply-connected but not homeomorphic to $\mathbb{R}^3$.

Moise's theorem:

Any topological 3-manifold has unique PL and smooth structures.

It has been discovered since the $1950s$ that higher-dimensional manifolds are easier to work with than 3 dimensional ones. The fundamental group plays an important role in all dimensions even when it is trivial, and relations between generators of the fundamental group correspond to two-dimensional disks, mapped into the manifold. In dimension 5 or greater, such disks can be put into general position

so that they are disjoint from each other, with no self-intersections, but in dimension 3 or 4 it may not be possible to avoid intersections, leading to serious difficulties.

◯ *4-dimensional manifolds*

In 1958, Russian Andrey **Markov**$(1903-1979)$ proved the result

Theorem:

The homeomorphism problem for closed 4 −manifold is algorithmically undecidable.

That means that there is no algorithm to classification closed 4 −manifolds. The groups isomorphism have a relation with spaces homeomorphisms. In 1982 , American **Freedman**$(1951-)$ managed to obtain a complete classification of closed oriented simply-connected topological 4 −manifolds using the Kirby−Siebenmann invariant. We have not a classification for smooth 4-dimetional manifolds. English Simon **Donaldson**$(1957-)$ who in 1983 did a work on smooth manifolds using bizarre connection with gauge theory. In fact he used gauge theory to investigate the manifold of dimension 4. He discovered that these spaces cannot have smooth structures and also the exotic sphere for the dimension 4.

We know that the topological Poincaré conjecture is true in all dimensions. There are problems around the fourth case.

Theorem:

The PL Poincaré conjecture is true for n −dimentional manifolds except possibly when $n=4$.

The differentiable Poincaré conjecture is more difficult, we know that it is not true for all dimensions.

Theorem:

The differential Poincaré conjecture is true for $n=1,2,3,5,6,12$ and 61.

Particularly, we do not know the case $n=4$. It is conjectured that it is false for all other dimensions.

◯ *Jordan Curve Theorem:*

Let in the plan a curve which is *simple*(without intersection) and close. This is called *Jordan curve*. In 1887 , French Camille **Jordan**$(1838-1922)$ stated the very evident intuitive result

Jordan curve theorem:

Every Jordan curve divides the plan in two region interior and exterior.

Fig. A loop a Jordan curve

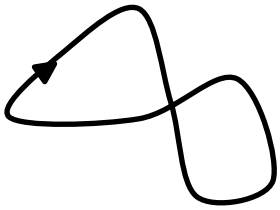


Fig. a loop which is not a Jordan curve.

He gave a proof that has not satisfied mathematicians[25]. Another proof was given by mathematician Oswald **Veblen**$(1880-1960)$ in 1905. A new curve the by Italian **Peano**$(18-19)$ in 1890 a space-filling curve, and also by David **Hilbert**$(18-19)$ in 1891. Such a curve passes by every point in unit square (in $\mathbb{R}^2$). It is not injective.

Exercises

1) Does the Jordan theorem hold on a torus?

g

Knots

[25] See Thomas **Hales**: "*Jordan's proof of the Jordan curve theorem*"(2007).

Another exiting chapter of algebraic topology is the theory of knots. A *Knot* is a closed loop. A trivial knot is a circle, also called an *unknot*. The simplest nontrivial knot is the *trefoil.*

Fig. The trefoil.

The principal problem on knot theory is When two knots are the same . Knot theory is considering is the study of embeddings of the circle in 3 −space.

The theory is old and one of the earliest papers on the subject was " *Remarks on the problem of position*" by French Alexandre **Vandermonde** $(1735-1796)$ in 1771, where he disserted on knots as a subject not concerned with measurement but on position. In 1876 Lord **Kelvin**$(18-19)$ believed that knot could be theory for atoms. Scottish mathematician Peter **Tait**$(1831-1901)$ then composed a list of all knots and published a tabulation of them in 1877. The task was to distinguish different knots from each other, meaning are not equivalent. This raised the problem of classification of knots in a definite list. It is known that the list is *infinite*. **Tait** made three conjectures, proved only between 1987 and 1991.

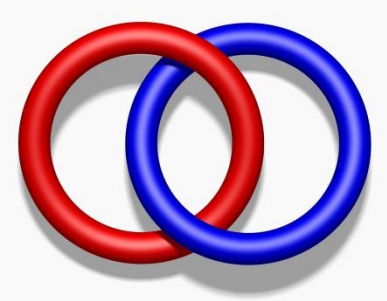
◯ *Oriented knot:*

In what follows we will note a knot by the symbol K. as for surfaces a and manifolds in general, we have to take into account the *orientation* of a knot. A right handed trefoil and the left handed trefoil are not equivalent. A knot has crossing and we perform a projection on the plan.

We can compose two knots K_1 and K_2 to get another knot : $K_1 \sharp K_2 = K_3$. The circle is noted K_0 and we have $K \sharp K_0 = K$. A knot K_1 that cannot be decomposed into two nontrivial knots $K_1 \neq K_2 \sharp K_3$ is a *prime* knot , an example of prime knots is the trefoil . In 1949, German Horst **Schubert**$(1919-2001)$ showed that every oriented knot in S^3 can decompose in a sum of prime knots.

◯ *Links:*

We can also construct shapes similar to knots called links. A *link* is What is not is an unlink.



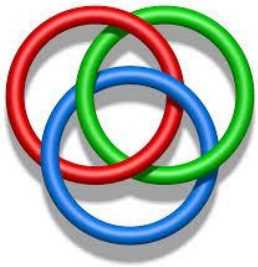


Fig. the Hopf link and the Borromeans rings.

◯ Knot Invariants :

Knots can be very complicated, and the main in this domain is to distinguish non-equivalent knots. But how to tell that a knot is different from another ? the idea to distinguish between two knots , as in the case of spaces, is to find ***invariants***. A knot when it itself is called a ***crossing*** the trefoil knot has three crossing. An interesting task is to tell if an untangled knot is the unknot. This is very difficult and it has been shown that it is in $co-P$ problems[26]. German Wolfgang **Haken**$(1928-2022)$ wrote a program of 130 pages to say if a knot is the trivial knot. The question is still open, searchers to programs to find between two knots but.. The first invariant

◯ ***Alexander polynomial:***

American James **Alexander** $(1888-1971)$ found an invariant, it is the knot polynomial in 1928. For the trefoil $\Delta_K = x^{-1} + x - 1$, For the unknot $\Delta_K = 1$. It is symmetric

$$\Delta_K(x) = \Delta_K\left(\frac{1}{x}\right)$$

Another is upon Alexander work is ***Conway polynomial*** noted ∇_K.

[26] In complexity theory. See *Computers and Intractability* by Michael **Garey**$(1945-)$ and *Introduction to the Theory of Computation* by Michael **Spiser**$(1954-)$. The Clay Institute announced 1 million dollars in 2000 to solve the $P = NP$ problem.

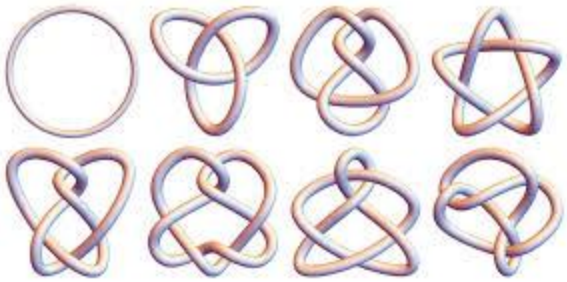
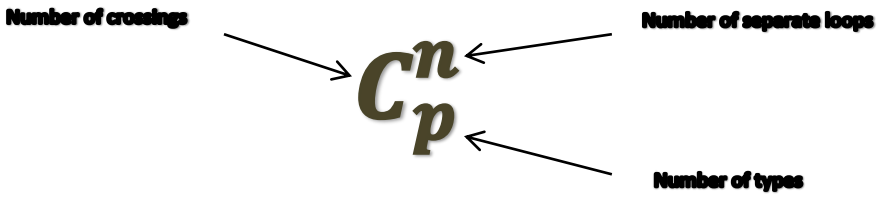


Fig. some knots

◯***Crossing number:***

Another invariant comes from the number of crossings in a knot. Crossings are the points where the knot crosses itself. A circle has the crossing number 0. Since the unknot have no crossing, so a knot has to have more than 1 crossing to be nontrivial. So a knot can have more than 2 crossings. We introduce the Alexander—Briggs notation



So a knot with 3 crosses is noted 3_1 , it is the trefoil knot. So crossing number as an invariant which is defined as the smallest number of crossings that a knot has.

In 1956, German Horst **Schubert** classified a type of knots in $\mathbb{R}^3$ named "knots with two bridges". In 1974 an American lawyer, Kenneth **Perko**$(1943-)$, discovered in the known list that are two knots that are the same.

◯ ***Jones polynomial:***

In 1984 , New Zealand mathematician Vaughan **Jones**$(1952-2020)$ found a new invariant: the *Jones polynomial* $P_J(K)$. His polynomial may have negative exponent(like the Laurent polynomial in complex analysis) . we have first for the unknot： $P_J(S^1)=1$, and in general

$$\sqrt{t}P_1(t)-tP_2(t)=(\sqrt{t}-\frac{1}{\sqrt{t}})P_0(t)$$

it is stronger than Alexander polynomial. Five years later and in 1989, a result due to American **Gordan**$(1945-)$ and **Lueke**$(19-)$states that a knot K is determined by its complement, that is:

Theorem

Two knots K_1 and K_2 are equivalent if and only if their complements $\mathbb{R}^3 - K_1$ and $\mathbb{R}^3 - K_2$ are homeomorphic

◯ ***The hyperbolic volume:***

Another with the complement can be as an invariant. We define the *hyperbolic volume* of a knot K(or a link in general) as the volume of its complement. It by the integral and a real number.

◯ Braids :

A braid is

Exercise

1) Show that there is no 1-crossing unknot.
1) Show that there are no 2-crossing unknots .

Open Problems

I started to gather some open problems in topology around manifolds. These can be differential (differential topology) or algebraic. Then my eyes fell on a list of 80 problems in low-dimensional topology. The list was posed by American Robion **Kirby**$(1938-)$ in 1978 and then has been updated in 1995. On October 30^{th} , 2023 a workshop began in Pasadena(California) to compose a third new updated list. Previously and in 1963, American John **Milnor**$(1931-)$ proposed list of 7 problems in topology including the Poincaré conjecture , some of them are still open. Another reference is *Open Problems in Topology*(1994) edited by Jan van **Mill**$(1951-)$.

– The inscribed square problem[27].

– Classification of manifolds when $n = 3$.

– Classification of manifolds when $n = 4$.

– The Redshift conjecture .

– What are $\pi_n(S^2)$ for $n \geq 1$?

– The volume conjecture.

– The smooth Poincaré conjecture.

– The PL Poincaré conjecture for $n > 6$.

– Are[28] there exotic 4 –spheres ?

– Is Burau representation faithful for $n = 4$?

– The unknotting problem.

– Can all topological manifolds be triangulated?

– The Borel conjecture[29].

– Is the crossing number additive in the knot sum ?

– Is the Jones polynomial an unknot detector?

– Milnor conjecture for dimensions $n = 4, 5$.

– The kissing number problem.

– The Novikov conjecture[30].

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[27] By German mathematician Otto **Toeplitz**($1881 - 1940$) in his paper «*Über einige Aufgaben der Analysis situs* « (1911).

[28] The *smooth Poincaré conjecture* says that there are no such spheres. See article: "*Man and machine thinking about the smooth 4-dimensional Poincaré conjecture*".

[29] Due to French mathematician Armand **Borel**($1923 - 2003$) in 1953.

[30] Due to Russian mathematician Sergei **Novikov**($1938 - 2024$) in 1965.